Math 2143 - Brief Calculus with Applications

Test #1 - 2013.02.14

Solutions

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1. Compute the following limits.

(a) $\lim_{x\to 2} x^2+4x-8$

Since this is a polynomial, we can simply plug in the limit! Thus, we have

$$\lim_{x\to 2} x^2 + 4x - 8 = 4 + 8 - 8 = 4$$

(b) $\lim_{x\to 1} \dfrac{x^2+2x-3}{x+3}$

This is a rational function, also, it is not zero in the denominator, thus we can plug in the limit yet again!

$$\lim_{x\to 1} \frac{x^2+2x-3}{x+3} = \frac{0}{4} = 0$$

(c) $\lim_{x\to 2} \dfrac{x+3}{x^2+2x-3}$

Here we have to do some factoring first:

$$\begin{aligned}
\lim_{x\to 2} \frac{x+3}{x^2+2x-3} &= \lim_{x\to 2} \frac{x+3}{(x+3)(x-1)} \\
&= \lim_{x\to 2} \frac{x+3}{x+3}\frac{1}{x-1} \\
&= \lim_{x\to 2} \frac{1}{x-1} \\
&= 1
\end{aligned}$$

(d) $\lim_{x\to 1^+} \dfrac{|x-1|}{x-1}$

Using $x > 1$ in the limit, we need to get rid of the absolute values before computing the limit:

$$\begin{aligned}
\lim_{x\to 1^+} \frac{|x-1|}{x-1} &= \lim_{x\to 1^+} \frac{x-1}{x-1} \\
&= \lim_{x\to 1^+} 1 = 1
\end{aligned}$$

(e) $\lim_{x\to 1^-} \frac{|x-1|}{x-1}$

Using $x < 1$ in the limit, we need to get rid of the absolute values before computing the limit:

$$\begin{aligned}\lim_{x\to 1^+} \frac{|x-1|}{x-1} &= \lim_{x\to 1^+} \frac{-(x-1)}{x-1} \\ &= \lim_{x\to 1^+} -1 = -1\end{aligned}$$

2. Compute the following derivatives.

(a) $\frac{d}{dx}\left(x^3 - 4x^2 + 2\right)$

This is the simply the power rule with the sum/difference rule

$$\frac{d}{dx}\left(x^3 - 4x^2 + 2\right) = 3x^2 - 8x$$

(b) $\frac{d}{dw}\left(w^3 - \frac{4}{w^2} + 2\right)$

Once again, the power rule with the sum/difference rule

$$\frac{d}{dw}\left(w^3 - \frac{4}{w^2} + 2\right) = 3w^2 + \frac{8}{w^3}$$

(c) $\frac{d}{dz}\left(\sqrt{z^3 - 4z^2 + 2}\right)$

Finally, we get to employ the chain rule:

$$\begin{aligned}\frac{d}{dz}\left(\sqrt{z^3 - 4z^2 + 2}\right) &= \frac{1}{2\sqrt{z^3 - 4z^2 + 2}} \cdot \frac{d}{dz}\left(z^3 - 4z^2 + 2\right) \\ &= \frac{1}{2\sqrt{z^3 - 4z^2 + 2}} \cdot \left(3z^2 - 8z\right)\end{aligned}$$

(d) $\frac{d}{dx}\left(\frac{x^3 - 4x^2}{3x^2 - 2\sqrt{x} - 1}\right)$

Lastly, we use the quotient rule on this problem:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^3 - 4x^2}{3x^2 - 2\sqrt{x} - 1}\right) &= \frac{\frac{d}{dx}\left(x^3 - 4x^2\right) \cdot \left(3x^2 - 2\sqrt{x} - 1\right) - \left(x^3 - 4x^2\right) \cdot \frac{d}{dx}\left(3x^2 - 2\sqrt{x} - 1\right)}{\left(3x^2 - 2\sqrt{x} - 1\right)^2} \\ &= \frac{\left(3x^2 - 8x\right) \cdot \left(3x^2 - 2\sqrt{x} - 1\right) - \left(x^3 - 4x^2\right) \cdot \left(6x - \frac{1}{\sqrt{x}}\right)}{\left(3x^2 - 2\sqrt{x} - 1\right)^2}\end{aligned}$$

3. Find the equation of the tangent line to the function $f(x) = \sqrt{4x^2 - 2x + 1}$ at $x = \frac{1}{2}$.

We use the formula $y - y_0 = m(x - x_0)$, where slope is $f'(x_0)$. So we fill in the blanks, $x_0 = \frac{1}{2}$, $y_0 = 1$. Next we need to compute the derivative, which is

$$f'(x) = \frac{8x - 2}{2\sqrt{4x^2 - 2x + 1}} = \frac{4x - 1}{\sqrt{4x^2 - 2x + 1}}$$

Plugging in x_0 gives $f'\left(\frac{1}{2}\right) = 1$. So the equation of our tangent line is $y - 1 = x - \frac{1}{2}$.

4. Sketch the graph of the function $r(x) = \dfrac{(3x-2)(x+4)}{(2x+1)(x-2)}$ based on calculations of asymptotes and intercepts.

There are vertical asymptotes at $x = -\frac{1}{2}$ and $x = 2$. The horizontal asymptote is at $y = \frac{3}{2}$. The limits at the vertical asymptotes are very straightforward to compute, and can be determined from the graph below.

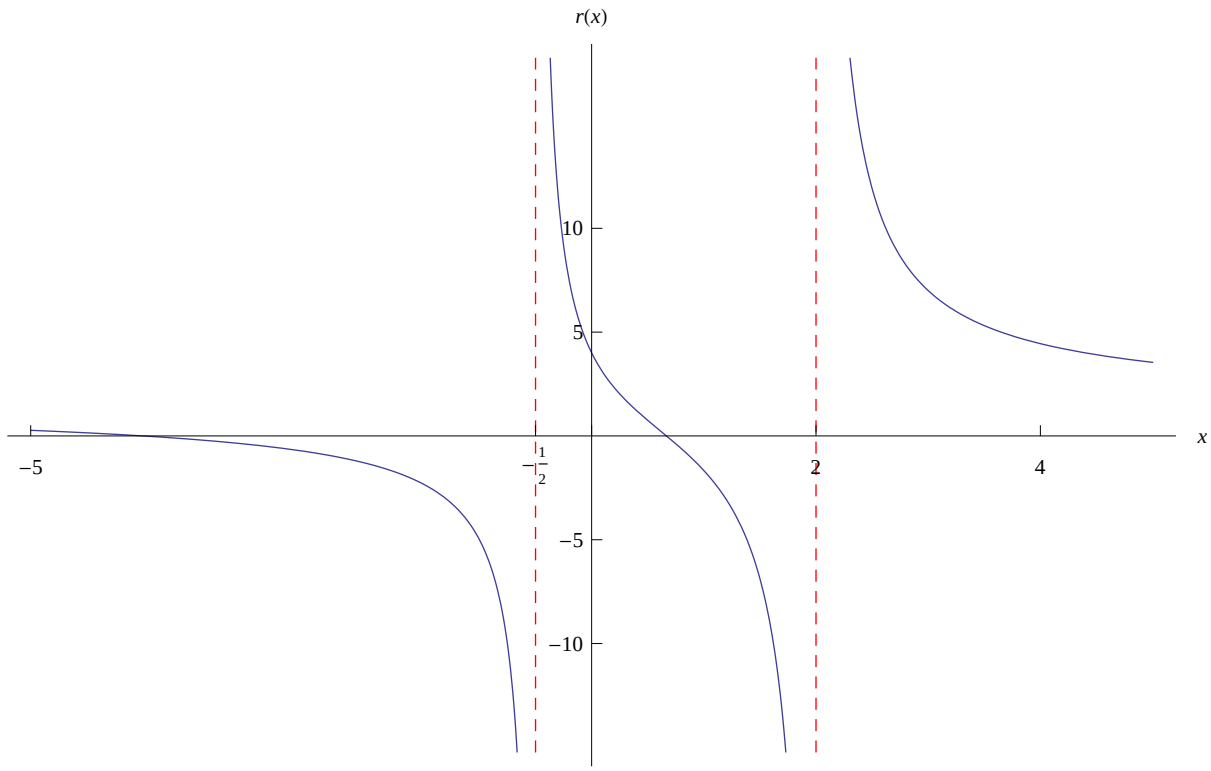

Figure to go along with problem 4.

5. Sketch the graph of the function $f(x) = x^3(x-1)$. Before doing so, compute all of the following to help you in graphing the function: (a) the roots, (b) y-intercept, (c) intervals of increase and decrease, (d) critical points, (e) intervals of concavity, and (f) inflection points.

(a) Roots are $x = 0$ and $x = 1$.

(b) The y-intercepts is at $y = 0$, thus the function passes through the origin.

(c) To compute this, we need $f'(x)$, we can use the product rule, then simplify:

$$f'(x) = 3x^2(x-1) + x^3 = x^2(4x-3)$$

Notice that x^2 is always positive, and thus, the sign of $f'(x)$ is determined only by the $4x-3$ term, so $f'(x) > 0$ for x on the interval $(3/4, \infty)$, and decreasing on the intervals $(-\infty, 0)$ and $(0, 3/4)$.

(d) The critical points are at $x = 0$ and $x = 3/4$.

(e) Next, we compute $f''(x)$, which requires the product rule or an expansion:

$$f''(x) = 2x(4x-3) + x^2 \cdot 4 = 6x(2x-1)$$

Intervals of concavity are easy to compute here. Since $f''(x)$ is parabola opening upward, $f''(x) < 0$ ($f(x)$ concave down) on $(0, 1/2)$ and similarly, $f(x)$ is concave up on $(-\infty, 0)$ and $(1/2, \infty)$.

(f) There are two inflection points here, where the concavity changes. They are $x = 0$ and $x = 1/2$.

Now we have enough information to graph!

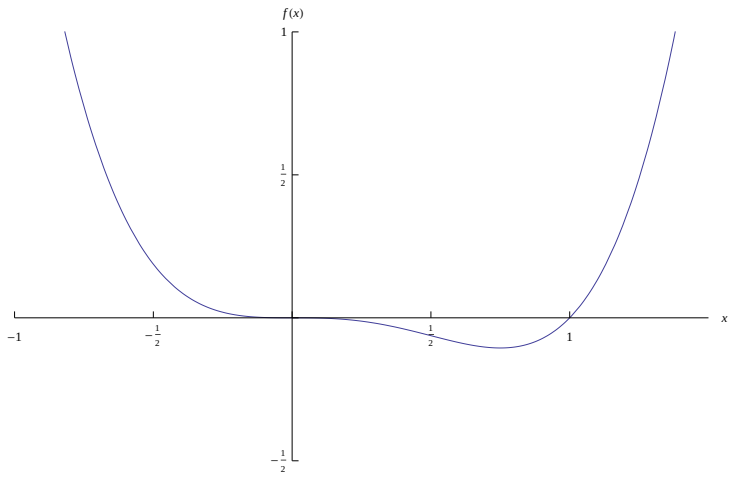

Figure to go along with problem 4.

6. Approximate $\sqrt[3]{26}$ as a fraction using the tangent line to the function $f(x) = x^{1/3}$ at $x = 27$.

We simply need to compute the tangent line at $x = 27$ for $f(x)$, and then plug the value $x = 26$ into it. The tangent line equation was given in a previous problem, so we simply compute $f'(x) = \frac{1}{3x^{2/3}}$, and plug in $x = 27$ to get $f'(27) = \frac{1}{27}$. So our tangent line is $y - 3 = \frac{1}{27}(x - 27)$. Plugging in the value $x = 26$ gives $\sqrt[3]{26} \approx 3 - \frac{1}{27} = \frac{60}{27}$.

7. You are filling up a can with water. The can is cylindrical in shape, with diameter 6 cm. At what rate is the volume of water in the can increasing if the height of the water in the can is increasing by 1 cm/sec and is at a height of 10 cm in the can?

The volume of the water in the can, which is a function of time, depends on the radius $r(t)$ and height $h(t)$ of the water, in particular:

$$V(t) = \pi r(t)^2 \cdot h(t)$$

Note that in this case, $r(t) = 6$ cm for all time, thus $V(t) = 36\pi\, h(t)$. Taking a time derivative gives $V'(t) = 36\pi\, h'(t)$. We can now plug in the given values (for when the height is 10 cm): $V'(t) = 360\pi$ cm^3.