Math 2283 - Introduction to Logic
Final Exam

Assigned: 2012.11.26

Due: 2012.12.10 at noon

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire, but you do have to attach the end of exam survey questions. Remember, you have two whole weeks to work on this, and it will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

Printed Name: ____________________________________

Signature: ____________________________________

Date: ____________________________________

1. Consider the following definition of $\prec$:

$$x \prec y \Leftrightarrow |x| < |y|$$

Also, we will define $x \succ y$ by $y \prec x$. The set of real numbers, together with the definitions of $\prec$ and $\succ$ do NOT constitute a model of the axiomatic system found on page 156. Determine which of the axioms hold, and which do not. Be sure to give examples where axioms do not hold. Is there a subset of the real numbers for which we do have a model for the axiomatic system?

2. Determine if the relation $\prec$ on the set of real numbers has any of the following properties: reflexive, irreflexive, symmetric, asymmetric, transitive, intransitive and connected.

3. Define the relation R as follows: $xRy \Leftrightarrow x = y + 1$.

(a) Is the relation $R/>$ reflexive?

(b) Is the relation $R/<$ reflexive?

(c) Is the relation $R/>$ symmetric?

(d) Is the relation $R/<$ symmetric?

4. Construct a truth table for the following sentence: $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (q \vee r)$.

5. Start with statement $[p \rightarrow (q \vee r)]$, and derive the equivalent statement $[(p \rightarrow q) \vee (p \rightarrow r)]$ using only logical manipulations. You are not allowed to assume anything except the entire first statement, nor are you allowed to use a truth table.

6. For this problem, you are to prove the closed system property for three conditional statements. Assume the following:

$$p_1 \rightarrow q_1,\ p_2 \rightarrow q_2,\ p_3 \rightarrow q_3$$

along with

$$p_1 \vee p_2 \vee p_3,\ q_1 \rightarrow \sim q_2,\ q_2 \rightarrow \sim q_3,\ q_1 \rightarrow \sim q_3.$$

Prove the following statements can be derived from the above assumptions:

$$q_1 \rightarrow p_1,\ q_2 \rightarrow p_2,\ q_3 \rightarrow p_3$$

7. Determine which of the following quantified sentences are true. If a sentence is false, give a counterexample to justify your answer. You may assume that the universe of discourse is the set of real numbers.

(a) $\underset{x,y}{\mathbf{E}} \left[(x^2 + y^2 = 1) \wedge (x = y)\right]$

(b) $\underset{x,y}{\mathbf{E}} \left[(x^2 + y^2 = 1) \wedge (x = -y)\right]$

(c) $\underset{x,y_1}{\mathbf{E}} \underset{y_2}{\mathbf{A}} \left[(x^2 + y_1^2 = 1) \wedge (x^2 + y_2^2 = 1 \rightarrow (y_2 = y_1))\right]$

(d) $\underset{x}{\mathbf{A}} \underset{y_1,y_2}{\mathbf{E}} \left[|x| < 1 \rightarrow \left[(x^2 + y_1^2 = 1) \wedge (x^2 + y_2^2 = 1) \wedge (y_2 \neq y_1)\right]\right]$

8. Using the definitions of $\prec$ and $\succ$ from problem 1, determine which of the following quantified sentences are true. If a sentence is false, give a counterexample to justify your answer. You may assume that the universe of discourse is the set of real numbers.

(a) $\underset{x}{\mathbf{A}}\,\underset{y}{\mathbf{E}} \sim (x \prec y)$

(b) $\underset{x}{\mathbf{E}}\,\underset{y}{\mathbf{A}} \sim (x \prec y)$

(c) $\underset{x,y}{\mathbf{A}}\,[(x \neq y) \rightarrow ((x \succ y) \vee (x \prec y))]$

(d) $\underset{x,y}{\mathbf{A}}\,\underset{z}{\mathbf{E}}\,[(x \prec y) \rightarrow ((x \prec z) \wedge (z \prec y))]$

9. Prove the following theorem:

$$p \rightarrow [\ (p \rightarrow q) \rightarrow q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

Theorem I: $[\ p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$
Theorem II: $p \rightarrow p$

10. Is the relation $\prec$ as defined in problem 1 monotonic with respect to the standard definition of addition? What about multiplication? Does anything change if the universe is the set of all non-zero real numbers?

11. Given the definitions of $\prec$ and $\succ$ as in problem 1, and the universe to be the set of real numbers, determine the following:

(a) $\prec \cap <$

(b) $\prec \cup <$

(c) $\prec \cap (\succ)'$

Essay Question: Write a one page essay (typed, single spaced) which utilizes the themes and topics of Chapter IX to relate *Gödel's First Incompleteness Theorem*, *Peano Axioms*, and the concept of *finitism* in logic and arithmetic.

Survey Questions

Please take the time to answer the following questions (and answer them seriously). You will not be counted off for not answering the questions, nor will your answer in any way affect your grade, so be honest!

1. What advice would you give a student taking this course next year under the assumption that the same book will be used?

2. What did you like most about this class?

3. What did you like least, besides the book or the amount of material covered, about this class?

4. What did you expect to get out of this class before the semester began (and I do not mean grade-wise here)?

5. What do you feel is the most important concept/idea that you have learned from this course?

6. Do you feel that the instructor could have been more helpful outside of class somehow? If so, how?

7. What do you expect to receive as a final grade, and what do you hope to get?