

Möbius Transformations

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1 Introduction to Compositions

So remember that we have two functions, T and S , along with their inverses, T^{-1} and S^{-1} respectively. As we travel through the tree, we will be composing these functions with each other countless times in varying arrangements. Anyone who has taken a college algebra course knows how difficult this can be. For instance, if $f(x) = \sqrt{x^2 + 1}$ and $g(x) = \frac{x}{x+1}$, then we have the following two compositions:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = \sqrt{g^2(x) + 1} = \sqrt{\left(\frac{x}{x+1}\right)^2 + 1}, \\(g \circ f)(x) &= g(f(x)) = \frac{f(x)}{f(x) + 1} = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1} + 1}\end{aligned}$$

First off, note that in general, $f(g(x)) \neq g(f(x))$, and that even the simple functions defined above create complicated compositions. Try to compute $f \circ g \circ g \circ f$ by hand. Furthermore, imagine if we have to compose 2000 functions together!

2 Möbius Transformations

As we learned in lecture, the functions T and S which we will be using belong to a class of complex valued functions known as Möbius transformations. Any Möbius transformation can be expressed in the following form:

$$M(z) = \frac{az + b}{cz + d} \tag{1}$$

where z is a complex valued variable, and the scalars a , b , c and d are complex constants such that $ad - bc \neq 0$. If we wish to compose two Möbius transformations M_1 and M_2 , we end up with another Möbius transformation! This may seem counterintuitive, but take a look at the following algebraic maneuvers:

$$\begin{aligned}(M_1 \circ M_2)(z) &= M_1(M_2(z)) \\&= \frac{a_1 M_2(z) + b_1}{c_1 M_2(z) + d_1} \\&= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\&= \frac{\frac{a_1 a_2 z + a_1 b_2}{c_2 z + d_2} + b_1}{\frac{c_1 a_2 z + c_1 b_2}{c_2 z + d_2} + d_1} \\&= \frac{\frac{a_1 a_2 z + a_1 b_2}{c_2 z + d_2} + b_1}{\frac{c_1 a_2 z + c_1 b_2}{c_2 z + d_2} + d_1} \cdot \frac{c_2 z + d_2}{c_2 z + d_2} \\&= \frac{a_1 a_2 z + a_1 b_2 + b_1 c_2 z + b_1 d_2}{c_1 a_2 z + c_1 b_2 + d_1 c_2 z + d_1 d_2} \\&= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)} \\&= \frac{a_3 z + b_3}{c_3 z + d_3}\end{aligned} \tag{2}$$

Here, we have set $a_3 = a_1 a_2 + b_1 c_2$, $b_3 = a_1 b_2 + b_1 d_2$, $c_3 = c_1 a_2 + d_1 c_2$ and $d_3 = c_1 b_2 + d_1 d_2$ are all complex constants since a_1 , a_2 , b_1 , b_2 , c_1 , c_2 , d_1 and d_2 are all complex constants. What this tells us, is that even if we end up composing 2000 Möbius transformations together, the form of the resulting function will look just like that from equation (1)!

3 Möbius Transformations and Matrix Multiplication

Even more impressive than the fact that the composition of any number of Möbius transformations yields yet another Möbius transformation is the fact that the compositions are easy to compute. But before we can do this, we need to take a look at matrix multiplication. Which in turn, means we need to look at matrices. A matrix is a collection of elements arranged in rows and columns. For our situation, we are concerned with matrices which have two rows and two columns. The following are examples of 2×2 matrices:

$$A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3+i & 6 \\ -2 & -1+i \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{2}+i & 3-\sqrt{2}i \\ \sqrt{5}+\sqrt{5}i & 6-6\sqrt{2}i \end{bmatrix}$$

The first matrix, A is has real valued (integer even) entries, while the matrices B and C are complex valued since each contains at least one entry with $i = \sqrt{-1}$ in it.

Basic arithmetic on 2×2 matrices consists of scalar multiplication, addition and subtraction, and can be done entry wise:

$$\begin{aligned} 2 \cdot A &= 2 \cdot \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 & 2 \cdot 3 \\ 2 \cdot (-4) & 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -8 & 2 \end{bmatrix} \\ A + B &= \begin{bmatrix} 2 + (-3+i) & 3+6 \\ -4 + (-2) & 1 + (-1+i) \end{bmatrix} = \begin{bmatrix} -1+i & 9 \\ -6 & i \end{bmatrix}, \\ B - C &= \begin{bmatrix} -3+i - (\sqrt{2}+i) & 6 - (3-\sqrt{2}i) \\ -2 - (\sqrt{5}+\sqrt{5}i) & -1+i - (6-6\sqrt{2}i) \end{bmatrix} = \begin{bmatrix} -3-\sqrt{2} & 3+\sqrt{2} \\ -2-\sqrt{5}-\sqrt{5}i & -7+(1-6\sqrt{2})i \end{bmatrix} \end{aligned}$$

The more complicated operations are multiplication and division. To multiply two square matrices together, we multiply rows of the left matrix by columns of the second matrix. We will do this for two arbitrary matrices M_1 and M_2 . The following formula works for any two 2×2 matrices.

$$\begin{aligned} M_1 \cdot M_2 &= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \end{aligned} \tag{3}$$

Now, we need to connect this with Möbius transformations. Firstly, if we look at equation (1), note that there are 4 constants, a , b , c and d . We can express ANY Möbius transformation as a 2×2 matrix with complex coefficients as follows:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4}$$

Even more importantly, if you compare the final expression in the computation of $M_1 \cdot M_2$ in (2) with the matrix product given in (3), you will see that the composition of Möbius transformations is simply the corresponding matrix multiplication of their 2×2 matrix representations! For those of you too lazy to look back at the end result of equation (2), we had

$$(M_1 \circ M_2)(z) = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}, \tag{5}$$

which in matrix form is

$$M_1 \cdot M_2 = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \tag{6}$$

So what are the implications of this result? If we need to compose 2000 Möbius transformations, we are simply multiplying 2×2 matrices together 2000 times. This in itself may require some computation time, but a subroutine can be written to easily multiply two matrices together.

The last operation we need to talk about is matrix division. Remember that division is simply multiplication by multiplicative inverse. We will not go into detail here, but given a matrix defined as (4), we define the inverse, M^{-1} as follows:

$$\begin{aligned} M^{-1} &= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \\ &= \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned} \tag{7}$$

So we now can also determine what the multiplicative identity is for 2×2 matrices. If we multiply M and M^{-1} as defined in (4) and (7) respectively, we see that

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1} \cdot M$$

We denote the 2×2 multiplicative identity matrix by I_2 , and as such, $M \cdot I_2 = I_2 \cdot M = M$ for any 2×2 matrix M . This is one example of a situation where multiplication is commutative for 2×2 matrices.

Do not be fooled, in general, $M_1 \cdot M_2 \neq M_2 \cdot M_1$, so order matters. However, we do get that matrix multiplication is associate, that is $(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3)$. Furthermore, be very careful, not every matrix has an inverse! In order for a matrix to have a multiplicative inverse, we require that $ad - bc \neq 0$ (this quantity is called the *determinant* of a 2×2 matrix). We will never encounter a situation where $ad - bc = 0$. This is due to the fact that we do not actually have a Möbius transformation in the case that $ad - bc = 0$.

4 Matrix representations of T and S

From the presentation Dr. Matthews gave, we were given that $T(z) = \mu + \frac{1}{z}$ and $S(z) = z + 2$. We are now in a position to apply our newfound knowledge of matrix representation of Möbius transformations to our program. First off, note that we can represent both $T(z)$ and $S(z)$ in standard Möbius transformation form as

$$T_\mu(z) = \frac{-i\mu \cdot z - i}{-i \cdot z + 0}, \quad S(z) = \frac{1 \cdot z + 2}{0 \cdot z + 1}, \quad (8)$$

which in turn gives

$$T_\mu = \begin{bmatrix} -i\mu & -i \\ -i & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad (9)$$

We now also can compute T^{-1} and S^{-1} using the definition given in (7) and applying it to (9). This gives

$$T_\mu^{-1} = \begin{bmatrix} 0 & i \\ i & -i\mu \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad (10)$$

It is easy to compute that $T \cdot T^{-1} = T^{-1} \cdot T = I_2 = S \cdot S^{-1} = S^{-1} \cdot S$. Remember, once again, that we cannot assume that matrix multiplication is commutative, so if we compute $T \cdot S \cdot T^{-1}$, this should not be equal to S for instance:

$$T_\mu \cdot S \cdot T_\mu^{-1} = \begin{bmatrix} 1 + 2\mu & -2\mu^2 \\ 2 & 1 - 2\mu \end{bmatrix},$$

$$T_\mu^{-1} \cdot S \cdot T_\mu = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Converting the above matrices back to function notation gives the two Möbius transformations:

$$(T_\mu \circ S \circ T_\mu^{-1})(z) = \frac{(1 + 2\mu)z - 2\mu^2}{2z + (1 - 2\mu)},$$

$$(T_\mu^{-1} \circ S \circ T_\mu)(z) = \frac{z}{2z + 1}$$

5 Differentiation of Möbius transformations

Remember the definition of $M(z)$ given in (1). We can compute the derivative of function, which after some simplification, gives

$$M'(z) = \frac{ad - bc}{(cz + d)^2} \quad (11)$$

Now, since our functions in matrix form have determinant equal to 1, we have that $ad - bc = 1$. One might raise the question: What about the composition of these functions and their inverses? The answer lies in the following two statements: (1) The determinant of a product is the product of determinants, and (2) the determinant of an

inverse is the reciprocal of the determinant of the original matrix. These two facts imply that $ad - bc = 1$ for any composition and means that we can now simplify our derivative of $M(z)$ to be

$$M'(z) = \frac{1}{(cz + d)^2} \tag{12}$$

This is quite a simplification. Now, remember that c and d are possibly complex constants, while z is a complex variable. So if we want to find the real part of $(M'(z))^2$, which is what we will need to for the series approximation, we will have to do some complex-number simplification.