Math 2283 - Introduction to Logic
Final Exam

Assigned: 2013.11.27

Due: 2013.12.11 at 13:00

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire, but you do have to attach the end of exam survey questions. Remember, you have two whole weeks to work on this, and it will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

Printed Name: ______________________________

Signature: ______________________________

Date: ______________________________

Definitions:

A relation R is *strongly connected* iff $\underset{x,y}{A}\ xRy \vee yRx$.

A relation S is *antisymmetric* iff $\underset{x,y}{A}\ xRy \wedge yRx \longrightarrow x = y$.

1. (10 pts) Prove that if a relation R is reflexive, then it is not asymmetric.

2. (10 pts) Prove that if a relation R is antisymmetric, then R' is connected.

3. (10 pts) Prove that if a relation R' is strongly connected, then R is asymmetric.

4. (10 pts) Prove for any three arbitrary relations R, S, and T, that $(R \cup S)/T = (R/T) \cup (S/T)$.

5. (20 pts) Let R, S, and T be arbitrary relations. Only one of the the following sentences is true. Determine, with proof, which one is true, and describe what fails in trying to prove the other sentence.

$$(R \cap S)/T \subset ((R/T) \cap (S/T)), \quad (R \cap S)/T \supset ((R/T) \cap (S/T))$$

6. (20 pts) Consider the relation C amongst real numbers defined as

$$xCy \longleftrightarrow x^2 + y^2 = 1$$

Determine if C is any of reflexive, irreflexive, symmetric, asymmetric, antisymmetric, transitive, intransitive or connected over the set of real numbers.

7. (15 pts) Prove the following theorem:

$$p \to [\ (p \to q) \to q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

Theorem I: $[\ p \to (q \to r)] \to [q \to (p \to r)]$
Theorem II: $p \to p$

8. (20 pts) Using Axioms I – IX, Definition 1, and Theorems I – XX from pages 141 – 143, prove

$$K \cup [L \cap (M \cup N)] = [(K \cup L) \cap (K \cup M)] \cup [(K \cup L) \cap (K \cup N)]$$

9. (10 pts) Construct a truth table for the sentence $[(p \vee q) \wedge (p \longrightarrow r)] \longrightarrow (q \vee r)]$.

10. (20 pts) Prove the following theorem directly.

$$[(p \vee q) \wedge (p \longrightarrow r)] \longrightarrow (q \vee r)]$$

11. (15 pts) Define the universe of elements $\mathbb{U}$ to be

$$\mathbb{U} = \left\{ 2^k \;\middle|\; k = \ldots - 2, -1, 0, 1, 2, \ldots \right\}$$

Furthermore, define the operation of multiplication $\cdot$ as usual. Prove that $(\mathbb{U}, \cdot)$ satisfies all the requirements of being an Abelian group.

12. (30 pts) Define Axiomatic systems ($\mathfrak{A}$) and ($\mathfrak{B}$) on a class K with operation R as follows:

<u>Axiomatic system $\mathfrak{A}$:</u>
Axiom $1^{\mathfrak{A}}$: The relation R is connected in the class K.
Axiom $2^{\mathfrak{A}}$: The relation R is asymmetric in the class K.
Axiom $3^{\mathfrak{A}}$: The relation R is transitive in the class K.

<u>Axiomatic system $\mathfrak{B}$:</u>
Axiom $1^{\mathfrak{B}}$: The relation R is connected in the class K.
Axiom $2^{\mathfrak{B}}$: $(xRy \wedge yRz \wedge zRt \wedge tRu \wedge uRv) \longrightarrow \sim vRx$

Recall that Axiomatic system $\mathfrak{A}$ implies that the relation R orders the class K. Prove that axiomatic systems $\mathfrak{A}$ and $\mathfrak{A}$ are equipollent. What does this imply about the definition of an ordering of a class K using a relation R?