Math 1613 - Trigonometry

Exam #5 - 2010.11.16

Solutions

Pythagorean Identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \qquad 1 + \cot^2(\theta) = \csc^2(\theta)$$

Sum and Difference Identities:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \qquad \cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \qquad \sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)} \qquad \tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Double-Angle Identities:

$$\cos(2A) = \cos^2(A) - \sin^2(A) \qquad \cos(2A) = 1 - 2\sin^2(A)$$

$$\cos(2A) = 2\cos^2(A) - 1 \qquad \sin(2A) = 2\sin(A)\cos(A)$$

$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

Product-To-Sum Identities:

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

Sum-To-Product Identities:

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Verify each of the following identities.

1.

$$\begin{aligned}\frac{1}{\sec(t)-1}+\frac{1}{\sec(t)+1} &= 2\cot(t)\csc(t)\\ \frac{\sec(t)+1+\sec(t)-1}{\sec^2(t)-1} &=\\ \frac{2\sec(t)}{\tan^2(t)} &=\\ \frac{2\cos^2(t)}{\cos(t)\sin^2(t)}\\ 2\cot(t)\csc(t) &=\end{aligned}$$

2.

$$\begin{aligned}\frac{\tan(x+y)-\tan(y)}{1+\tan(x+y)\tan(y)} &= \tan(x)\\ \tan((x+y)-y) &=\\ \tan(x) &=\end{aligned}$$

3.

$$\begin{aligned}\sin\left(\frac{\pi}{6}+x\right)+\sin\left(\frac{\pi}{6}-x\right) &= \sqrt{3}\cos(x)\\ 2\sin\left(\frac{\frac{\pi}{6}+x+\frac{\pi}{6}-x}{2}\right)\cos\left(\frac{\frac{\pi}{6}+x-\left(\frac{\pi}{6}-x\right)}{2}\right) &=\\ 2\sin\left(\frac{\pi}{6}\right)\cos(x) &=\\ 2\frac{\sqrt{3}}{2}\cos(x) &=\\ \sqrt{3}\cos(x) &=\end{aligned}$$

4.

$$\begin{aligned}\sin(4x) &= 4\sin(x)\cos(x)-8\sin^3(x)\cos(x)\\ &= 4\sin(x)\cos(x)\left[1-2\sin^2(x)\right]\\ &= 2\sin(2x)\cos(2x)\\ &= \sin(4x)\end{aligned}$$

5.

$$\begin{aligned}\sin(2x) &= \frac{2\tan(x)}{1+\tan^2(x)}\\ &= \frac{2\tan(x)}{\sec^2(x)}\\ &= 2\sin(x)\cos(x)\\ &= \sin(2x)\end{aligned}$$

6.

$$\begin{aligned}\tan(\theta)\sin(2\theta) &= 2-2\cos^2(\theta)\\ \tan(\theta)2\sin(\theta)\cos(\theta) &=\\ 2\frac{\sin(\theta)}{\cos(\theta)}\sin(\theta)\cos(\theta) &=\\ 2\sin^2(\theta) &=\\ 2(1-\cos^2(\theta)) &=\end{aligned}$$

7. Find $\sin(y)$ given that $\cos(2y) = -\frac{1}{3}$, with $\frac{\pi}{2} < y < \pi$.

We use the double angle identity:

$$\cos(2y) = 1 - 2\sin^2(y)$$

to arrive at the equation

$$-\frac{1}{3} = 1 - 2\sin^2(y).$$

Solving for y we get

$$\frac{2}{3} = \sin^2(y),$$

or

$$\sin(y) = \pm\sqrt{\frac{2}{3}}.$$

Since y is in quadrant II, we pick the positive:

$$\sin(y) = \sqrt{\frac{2}{3}}.$$

8. Find $\sin(x-y)$, $\cos(x-y)$ and what quadrant $x-y$ is located in, given that $\sin(x) = -\frac{1}{2}$, $\cos(y) = -\frac{2}{5}$ and x, y are angles located in quadrant III.

To arrive at $\sin(x-y)$ and $\cos(x-y)$, we will need to determine $\cos(x)$ and $\sin(y)$:

$$\cos^2(x) + \left(-\frac{1}{2}\right)^2 = 1 \to \cos(x) = \pm\frac{\sqrt{3}}{2}.$$

Here x is in quadrant III, so $\cos(x) = -\frac{\sqrt{3}}{2}$. Similarly, we have

$$\sin^2(y) + \left(-\frac{2}{5}\right)^2 = 1 \to \sin(y) = \pm\frac{\sqrt{21}}{5}$$

Here y is in quadrant III, so $\sin(y) = -\frac{\sqrt{21}}{5}$.

Using the formula $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$, we get

$$\begin{aligned}\cos(x-y) &= -\frac{\sqrt{3}}{2}\cdot -\frac{2}{5} + -\frac{1}{2}\cdot -\frac{\sqrt{21}}{5}\\ &= \frac{2\sqrt{3}+\sqrt{21}}{10}.\end{aligned}$$

Next,

$$\begin{aligned}\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y)\\ &= -\frac{1}{2}\cdot -\frac{2}{5} - -\frac{\sqrt{3}}{2}\cdot -\frac{\sqrt{21}}{5}\\ &= \frac{2-3\sqrt{7}}{10}.\end{aligned}$$

Since $\cos(x-y) > 0$ and $\sin(x-y) < 0$, $x-y$ must be in quadrant IV.