Math 2215 - Calculus 1

Homework #1 Solutions

Assigned - 2011.01.12

Textbook problems:

Section 1.1 - 1, 6, 11

Section 1.2 - 1, 2, 3, 7, 9, 15, 25, 33

Section 1.3 - 1, 2, 8, 10, 13, 26, 29, 31, 39, 40, 64, 65

Section 1.4 - 1, 7, 14, 22, 23, 25, 27, 33, 40, 50

Section 1.5 - 1, 4, 5, 14, 19, 23, 31, 35, 41, 43

Fun Problems:

1. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$ mathematically? Explain this result in terms of physical quantities and properties also.

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = +\infty$$

In physical quantity terms, the faster an object is moving, the heavier it becomes, and thus the closer to the speed of light an object travels, the more energy is required to increase the velocity of the object.

2. Also in relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find the limit as $v \to c^-$ mathematically. Explain this result in terms of physical quantities and properties also.

$$\lim_{v \to c^-} L_0 \sqrt{1 - v^2/c^2} = L_0 \sqrt{1 - 1} = 0$$

In physical quantity terms, the faster an object moved, the smaller in length it becomes. For instance, this may allow one to move an object around a corner not normally possible, if done with enough velocity!

3. A common equation used to determine gravitational time dilation is derived from the *Schwarzschild metric*, which describes spacetime in the vicinity of a non-rotating massive spherically-symmetric object. The equation is:

$$t_o = t_f \sqrt{1 - \frac{2GM}{rc^2}}$$

where t_o is the time perceived between events for person X near the object in question, and t_f is the time perceived be a person far away from the object in question (i.e. not under the influence of any massive object, including the one under consideration). Also, G is the gravitational constant, M is the mass of the object creating the gravitational field, r is the radial distance from person X to the center of the massive object, and c is the speed of light. Also, if we define the *Schwarzschild radius* as $r_0 = \frac{2GM}{c^2}$, the the time t_0 can be expressed as

$$t_0 = t_f\sqrt{1 - \frac{r_0}{r}}.$$

(Note that if a mass collapses so that its surface lies at less than this radial coordinate r_0, then the object exists within a black hole.) As person X travels closer to the massive object (i.e. as $r \to r_0^+$) how do they perceive time with respect to the person far away? Also, how does the person far away perceive time with respect to person X as person X gets closer and close to the massive object?

First, note that

$$\lim_{r\to r_0^+} \sqrt{1 - \frac{r_0}{r}} = 0,$$

therefore, for $r_0 - r = \varepsilon$ (i.e. $r = \varepsilon + r_0$), we have

$$t_0 = \sqrt{\frac{\varepsilon}{r_0 + \varepsilon}}\, t_f \approx \sqrt{\frac{\varepsilon}{r_0}}\, t_f.$$

This means for a fixed time passage for person X close to the massive object, the corresponding time elapsed as perceived by the person far away is small (on the order of $\sqrt{\frac{\varepsilon}{r_0}}$ times the time observed to have passed by person X. Notice that is person X approaches a distance of r_0 from the object (the event horizon) time will have appeared to have come to a stand still to the person observing far away (and hence person X will never reach a distance of r_0 from the object from the viewpoint of the person far away). Furthermore, from person X's point of view, time will pass very quickly for the person far away from the massive object.

4. Give an example of a function $f(x)$ such that $|f(x)|$ is continuous everywhere but $f(x)$ itself is discontinuous at at least one point.

The easiest function to pick is on such as the following:

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

Notice that $|f(x)| = 1$ which is continuous everywhere, but $f(x)$ itself is not continuous at $x = 0$.

5. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

First, notice that the second term's denominator is always nonpositive, and since $b > 0$,

$$\lim_{x\to 1^-} \frac{b}{x^3 + x - 2} = -\infty$$

Since we know that at $x = 1$ the first term is finite (with value $\frac{a}{2}$), we can conclude that

$$\lim_{x\to 1^-} \frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = -\infty. \tag{1}$$

The roots of the denominator for the first term are

$$\left\{-1, -\frac{1}{2} + \frac{1}{2}\sqrt{5}, -\frac{1}{2} - \frac{1}{2}\sqrt{5}\right\},$$

where the middle root has value approximately .6180339880. Furthermore,

$$\lim_{x\to(-\frac{1}{2}+\frac{1}{2}\sqrt{5})^+} \frac{a}{x^3 + 2x^2 - 1} = \infty,$$

and

$$\lim_{x\to(-\frac{1}{2}+\frac{1}{2}\sqrt{5})^+} \frac{b}{x^3+x-2} = \frac{2b}{3(-3+\sqrt{5})},$$

which is finite. Therefore, we have

(2)
$$\lim_{x\to(-\frac{1}{2}+\frac{1}{2}\sqrt{5})^+} \frac{a}{x^3+2x^2-1} + \frac{b}{x^3+x-2} = \infty.$$

Putting (1) and (2) together, we can invoke the Intermediate Value Theorem to state that there is a root to the function on the interval $\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5}, 1\right)$ since the function is continuous on the interval in question and positive at one end, negative at the other.

6. Find numbers a and b such that

$$\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$$

First, we manipulate the limit a wee bit:

$$\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = \lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} \cdot \frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2}$$
$$= \lim_{x\to 0} \frac{ax+b-4}{x\left[\sqrt{ax+b}+2\right]}$$

At this point, the limit will exist ONLY if $b = 4$. In this case, we have

$$\lim_{x\to 0} \frac{ax}{x\left[\sqrt{ax+4}+2\right]} = \lim_{x\to 0} \frac{a}{\sqrt{ax+4}+2}$$

This new function on the RHS of the above equality is continuous at $x = 0$, so we can plug in the limit:

$$\lim_{x\to 0} \frac{a}{\sqrt{ax+4}+2} = \frac{a}{4}$$

Setting this equal to 1 gives $a = 4$. Thus,

$$\lim_{x\to 0} \frac{\sqrt{4x+4}-2}{x} = 1.$$

7. Define $f(x)$ as follows:

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ x, & x \text{ is rational} \end{cases}$$

for what values of x is $f(x)$ continuous?

First, we show that $f(x)$ is continuous at $x = 0$. We can chose $\epsilon = \delta$ to get that

$$|f(x) - f(0)| = |x - 0|$$

and if $|x - \delta| < \delta$, this implies that $|f(x) - f(0)| < \varepsilon$, as required.

If we choose $x = a \neq 0$ to be rational, then $f(a) = a$. Then the continuity argument looks like:

$$|x - a| < \delta \to |f(x) - a| < \varepsilon$$

However, if x is irrational such that $|x - a| < \delta$, notice that $f(x) = 0$ and thus $|f(x) - a| = |a|$. So if we had chosen $\varepsilon < a$, no δ would work, and since ε is arbitrary (i.e. this has to work for all ε), the function cannot be continuous at any rational nonzero point.

A similar argument can be done for $x = a \neq 0$ irrational.

8. Construct a *Mathematica* function which performs a rudimentary bisection root-finding algorithm for locating a root of a function $f(x)$ on a specified closed interval $[a, b]$. You do not have to do any fancy graphics if you do not want.