Math 2215 - Calculus 1

Homework #2 Solutions

Assigned - 2011.01.24

Textbook problems:

Section 2.1 - 1-4 all, 7, 13, 17, 20, 25, 27, 31

Section 2.2 - 13-18 all, 23, 24, 25, 28, 41, 55, 57

Section 2.3 - 1, 6, 10, 18, 20, 23, 28, 41, 42, 45, 63

Section 2.4 - 1, 2, 5, 12, 16, 17, 24, 40, 43

Section 2.5 - 1, 3, 5, 10, 21, 22, 23, 35, 36, 39, 40, 43, 50

Section 2.6 - 3, 10, 15, 16, 20, 25, 28, 37, 38, 41, 43, 45, 46

Section 2.7 - 1, 3, 5, 9, 12, 16, 17, 20, 23, 29, 31, 34

Section 2.8 - 1, 6, 7, 9, 12, 17, 21, 23, 27, 39, 40, 41

Fun Problems:

1. The following figure shows a circle of radius 1 inscribed in the parabola $y = x^2$. Locate the center of this circle.

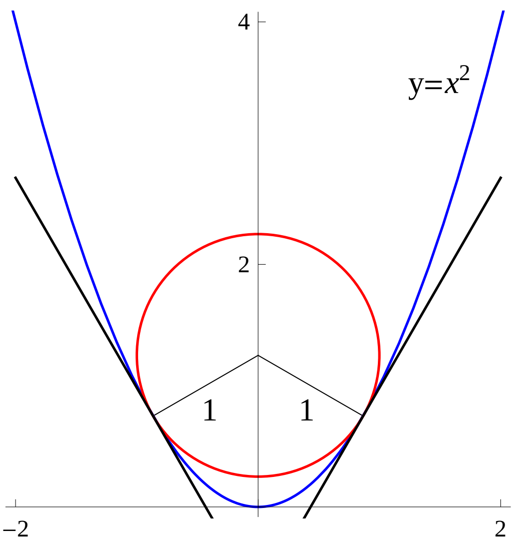

FIGURE 1. The lines of length 1 are perpendicular to the tangent lines

First note that the equation of the tangent line to $y = x^2$ at $x = a$ is given by

$$y = 2a\,(x - a) + a^2$$

where the slope m is $2a$ and the point is (a, a^2). The equation of the line perpendicular to the tangent line at $x = a$ passing through the same point is

$$y = -\frac{1}{2a}\,(x - a) + a^2$$

Plugging in $x = 0$ will correspond to the point where the perpendicular line intersects the y-axis, and would be the center of the circle. This gives the point to be $\left(0, \frac{1}{2} + a^2\right)$. Now the radius being 1 requires that the distance from the center, located at $\left(0, \frac{1}{2} + a^2\right)$, to a point on the circle (a, a^2), must be one. Thus

$$(a - 0)^2 + \left(\frac{1}{2} + a^2 - a^2\right)^2 = 1$$

Solving for a gives $a = \pm\frac{\sqrt{3}}{2}$. This gives the center of the circle to be at $\left(0, \frac{5}{4}\right)$.

2. There are exactly two points on the function $y = x^4 - 2x^2 - x$ which share a common tangent line. Find these two points. Be sure to graph the function and the tangent line you found.

First, let us call the two points in question (x_0, y_0) and (x_1, y_1). These have the same tangent line, thus

$$y = f'(x_0)(x - x_0) + f(x_0)$$
$$y = f'(x_1)(x - x_1) + f(x_1)$$

Setting these equal gives

$$f'(x_0)(x - x_0) + f(x_0) = f'(x_1)(x - x_1) + f(x_1)$$

but, $f'(x_0) = f'(x_1)$, so

$$f'(x_1)(x - x_0) + f(x_0) = f'(x_1)(x - x_1) + f(x_1)$$

or

$$f'(x_1)(x_1 - x_0) = f(x_1) - f(x_0)$$

Solving for $f'(x_1)$ gives

$$f'(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Actually plugging in values for the function gives the above equation to be

$$4x_1^3 - 4x_1 - 1 = \frac{\left(x_1^4 - 2x_1^2 - x_1\right) - \left(x_0^4 - 2x_0^2 - x_0\right)}{x_1 - x_0}$$

Also, since $f'(x_0) = f'(x_1)$ we get the following equation as well

$$4x_0^3 - 4x_0 - 1 = \frac{\left(x_1^4 - 2x_1^2 - x_1\right) - \left(x_0^4 - 2x_0^2 - x_0\right)}{x_1 - x_0}$$

We need to solve the following system of equations then

$$\begin{cases} 4x_0^3 - 4x_0 - 1 &= \frac{\left(x_1^4 - 2x_1^2 - x_1\right) - \left(x_0^4 - 2x_0^2 - x_0\right)}{x_1 - x_0} \\ 4x_1^3 - 4x_1 - 1 &= \frac{\left(x_1^4 - 2x_1^2 - x_1\right) - \left(x_0^4 - 2x_0^2 - x_0\right)}{x_1 - x_0} \end{cases}$$

So how do we solve these two equations? Well first we need two distinct solutions (thus not $x_1 = x_0$). We could graph the functions, but if we do, we need to know what to look for. First, as stated previously, we have to find solutions where $x_1 \neq x_0$, and thus on the graph, solutions that do not intersect $y = x$. Also, if there is a solution (a, b), then the graph should also exhibit an intersection at the point (b, a).

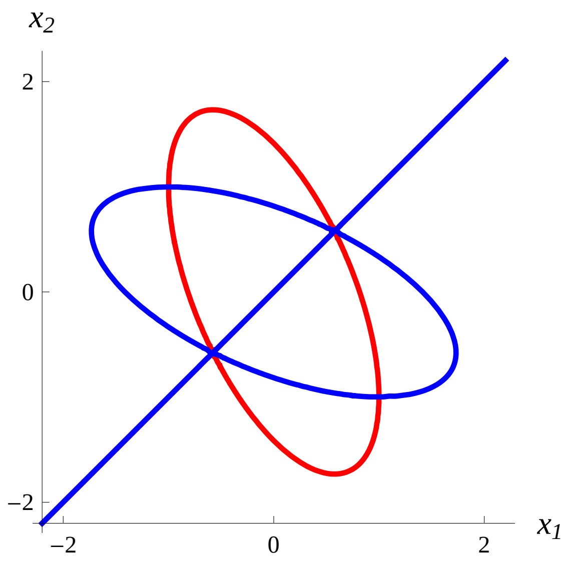

FIGURE 2. The implicitly defined functions intersect at two points off the $y = x$ line.

From the above figure, we see that the points we need are $x = 1$ and $x = -1$. Below is the graph of the function, and the tangent lines which is tangent at the two points $x = -1$ and $x = 1$. The equation for the tangent line is $y = -x - 1$.

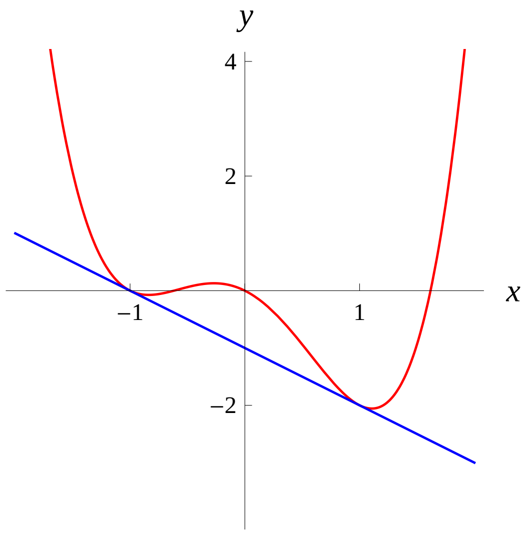

FIGURE 3. The function and tangent line at $x = -1$ and $x = 1$.

3. Consider the function:

$$g(x) = \begin{cases} x^2 - 2, & x \leq 1 \\ mx^2 + bx, & x > 1 \end{cases}$$

a) Find values for the constants m and b such that $g(x)$ has a tangent line at $x = 1$.

As before, we must set the slope of the tangent line from the left equal to the slope of the tangent line from the right. We know that $m_{tan^-} = 2$, so we need to compute m_{tan^+}:

$$\begin{aligned} m_{tan^+} &= \lim_{h\to 0} \frac{\big(m(1+h)^2 + b(1+h)2\big) - (m+b)}{h} \\ &= \lim_{h\to 0} \frac{m(1+2h+h^2) + b(1+h) - (m+b)}{h} \\ &= \lim_{h\to 0} \frac{m(2h+h^2)+bh}{h} \\ &= \lim_{h\to 0} \frac{h(m(2+h)+b)}{h} \\ &= \lim_{h\to 0} m(2+h)+b \\ &= 2m+b. \end{aligned}$$

Setting $m_{tan^-} = 2$ gives the equation $2 = 2m + b$. Solving for b gives $b = 2 - 2m$. We still do not have specific values for m and b though. Will any value of m work as long as $b = 2 - 2m$? Well we still have not shown that $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x)$. This we do next. Since the function is continuous from the left, we have that $\lim_{x\to 1^-} g(x) = -1$. The function is continuous from the right as well, assuming that m and b are fixed real numbers. So $\lim_{x\to 1^+} g(x) = m + b$. We now have the two equations:

$$b = 2 - 2m, \quad m + b = -1.$$

We can solve for both m and b in this two equation, two unknown system. We get $m = 3$ and $b = -4$.

b) Find the equation of the tangent line at $x = 1$ for the values of m and b found in part a).

Both curves go through the point $(1, -1)$, and the slope of the tangent line is $m = 2$. So we have the equation of the tangent line is $y + 1 = 2(x - 1)$. Notice that this simplifies to $y = 2x - 3$, which is the tangent line found in problem 2. This should obviously be the case!

c) Graph $g(x)$ about $x = 1$ and describe why the tangent line should exist for your specified value of m and b.

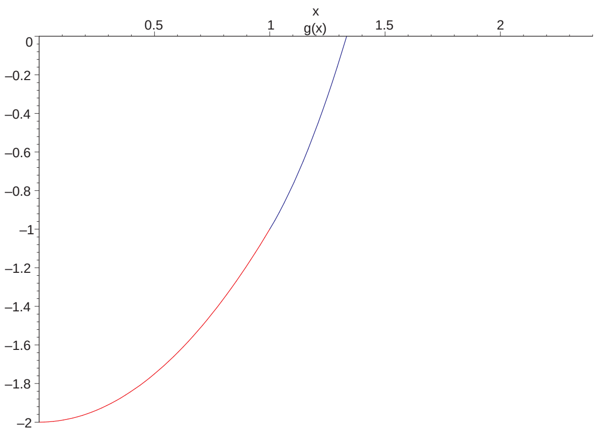

FIGURE 4. Graph of the piecewise function $g(x)$

It should be clear from the figure above that at $x = 1$, the slope of the tangent line from the left would equal the slope of the tangent line from the right.

4. Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^{1000} - 1}{x - 1}$$

Notice that if $f(x) = x^{1000}$, then $f(1) = 1$. Therefore, the above limit is simply $f'(1)$. By the power rule, $f'(x) = 1000x^{999}$, and thus $f'(1) = 1000$.

5. Find $f'(x)$ if $f(x) = |x|$. At what points is this function not differentiable?

We can rewrite $f(x) = \sqrt{x^2}$, and thus applying the chain rule, we get

$$f'(x) = \frac{2x}{2\sqrt{x^2}} = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

This function is equal to -1 for $x < 0$, and equal to 1 for $x > 0$, and is undefined at $x = 0$.

6. If $f(x) = (x-a)(x-b)(x-c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

First, we use the product rule twice:

$$f'(x) = 1 \cdot (x-b)(x-c) + (x-a) \cdot 1 \cdot (x-c) + (x-a)(x-b) \cdot 1$$

then dividing this by $f(x)$ gives

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} \\ &= \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\end{aligned}$$