# Math 2283 - Introduction to Logic
# Final Exam

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**Assigned**: 2010.11.23

**Due**: 2010.12.08 at 10:00 A.M.

**Instructions**: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire, but you do have to attach the end of exam survey questions. Remember, you have two whole weeks to work on this, and it will be graded accordingly.

**Agreement**: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

**Printed Name**: ________________________________

**Signature**: ____________________________________

**Date**: ________________________________________

1. Let $p \veebar q$ be the definition of *one and only one of* $p$ *and* $q$.

a) Construct a fundamental truth table for the connective $\veebar$.

b) Give a definition of $\veebar$ using the standard set of logical connectives.

c) Determine which of the following are logical laws via the method of truth tables

$$\begin{array}{llll} \text{I)} & ((p \veebar q) \wedge \sim q) \to p & \text{II)} & \sim ((p \veebar q) \to p) \leftrightarrow (q \wedge \sim p) \\ \text{III)} & ((p \veebar q) \wedge (q \veebar r)) \to \sim q & \text{IV)} & ((p \veebar q) \wedge (q \veebar r)) \to (\sim p \veebar r) \end{array}$$

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2. Prove the following theorem:

$$p \to [\ (p \to q) \to q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

Theorem I: $[\ p \to (q \to r)] \to [q \to (p \to r)]$
Theorem II: $p \to p$

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3. Prove the following theorem:

$$\text{if } x \leq y,\ y < z \text{ and } z \leq t, \text{ then } x < t,$$

using ONLY the following axiom:

$$\text{if } x < y \text{ and } y < z, \text{ then } x < z.$$

Here $(x \leq y) \leftrightarrow [(x < y) \vee (x = y)]$ is the standard definition of the $\leq$ symbol. You may assume all knowledge of logic, and hence may employ any shortcuts as needed. Be sure to state what laws/shortcuts you are using at each step in your proof.

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4. Determine which of the following quantified sentences are true. If a sentence is false, give a counterexample as proof. You may assume that the universe of discourse is the set of ***real numbers***.

a) $\underset{x}{\mathbf{A}}\ \underset{y}{\mathbf{E}} \left[\dfrac{1}{1+x^2} = y\right]$

b) $\underset{y}{\mathbf{A}}\ \underset{x}{\mathbf{E}} \left[\dfrac{1}{1+x^2} = y\right]$

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5. Determine which of the following quantified sentences are true. If a sentence is false, give a counterexample as proof. You may assume that the universe of discourse is the set of ***natural numbers***.

a) $\underset{x,y}{\mathbf{A}}\ \underset{z}{\mathbf{E}} [x > y \to y > z]$

b) $\underset{x,y}{\mathbf{A}}\ \underset{z}{\mathbf{E}} [x \geq y \to y \geq z]$

6. If we assume that the following two statements are true:

$$\begin{cases} p_1 \to q_1 \\ p_2 \to q_2 \end{cases}$$

it seems that the following statement should also hold:

$$(p_1 \wedge p_2) \to (q_1 \vee q_2)$$

Construct a proof that this is indeed the case. You MAY NOT use truth tables to show this.

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7. We start with the following set of axioms and definition (taken from Exercise 1 on page 140).

Axiom I: $K \subset K$
Axiom III: $K \cup L \subset M \leftrightarrow K \subset M \wedge L \subset M$
Axiom IV: $M \subset K \cap L \leftrightarrow M \subset K \wedge M \subset L$
Axiom VI: $K \subset \bigvee$
Axiom VII: $\bigwedge \subset K$
Definition I: $K = L \leftrightarrow K \subset L \wedge L \subset K$

Prove the following theorems:

a) Theorem XXI: $K \cup \bigwedge = \bigwedge \to K = \bigwedge$

b) Theorem XXII: $K \cap \bigvee = \bigvee \to K = \bigvee$

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8. Prove the following statements regarding relations:

a) If the relation $R$ is intransitive, then $R'$ is reflexive.

b) If the relation $R$ is irreflexive and transitive in a class, then $R$ asymmetrical.

c) If the relation $R$ is asymmetrical in a class, then $R$ is irreflexive.

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9. Consider the class $\mathbb{K}$ of possible sets formed by an arbitrary group of objects, the two relations $\subseteq$, $\supseteq$ and two operations $\cup$ and $\cap$. Are either of the given operations monotonic in the class $\mathbb{K}$ with respect to either of the relations $\subseteq$ and $\supseteq$?

Remember the definitions of the two relations $\subseteq$ and $\supseteq$ are

$$A \subseteq B \leftrightarrow \underset{x}{\mathbf{A}}\ (x \in A \to x \in B) \text{ and } A \supseteq B \leftrightarrow \underset{x}{\mathbf{A}}\ (x \in B \to x \in A)$$

respectively with $A, B \in \mathbb{K}$. Also, do not forget the universal and null sets.

10. For this problem, you are to prove the closed system property for three conditional statements. Assume the following:

$$p_1 \to q_1,\ p_2 \to q_2,\ p_3 \to q_3$$

along with

$$p_1 \vee p_2 \vee p_3,\ q_1 \to \sim q_2,\ q_2 \to \sim q_3,\ q_1 \to \sim q_3.$$

Prove the following statements can be derived from the above assumptions:

$$q_1 \to p_1,\ q_2 \to p_2,\ q_3 \to p_3$$

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11. Given the set $K = \{0, 1, 2, 3\}$, define two distinct operations, $\oplus$ and $\oslash$, on the set $K$ such that $K$ is an Abelian group with respect to $\oplus$, and also with respect to $\oslash$. Your definitions of the operations $\oplus$ and $\oslash$ must be distinct when applied to $K$.

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**Essay Question**: Write a one page essay (typed, single spaced) which utilizes the themes and topics of Chapter IX to relate *Gödel's First Incompleteness Theorem*, *Peano Axioms*, and the concept of *finitism* in logic and arithmetic.

# Common Laws of Logic

The following is a list of laws that we have used a decent amount in class or are just silly and may prove useful for this exam:

**Law of And (breaking apart)**: If you have the following statement:

$$P \wedge Q$$

then by using the following logical law

$$(P \wedge Q) \to Q,$$

and applying R.O.D. you get $Q$.

**Law of And (joining together)**: If you have the following two statements:

$$\begin{cases} P \\ Q \end{cases}$$

then by using the following logical law

$$P \to (Q \to (P \wedge Q))$$

and applying R.O.D. twice, you get $(P \wedge Q)$.

**Law of Or (single)**: If you have the following statement:

$$P$$

then by using the following logical law

$$P \to (P \vee Q)$$

and applying R.O.D. you get $(P \vee Q)$.

**Law of Or (double)**: If you have the following two statements:

$$\begin{cases} P \\ Q \end{cases}$$

then by using the following logical law

$$P \to (Q \to (P \vee Q))$$

and applying R.O.D. twice, you get $(P \vee Q)$.

**Law of Or (repeat)**: If you have the following statement:

$$P \vee P$$

then by using the following logical law

$$(P \vee P) \to P$$

and applying R.O.D., you get just $P$.

**Law of Conditional**: If you have the following two statements:

$$\begin{cases} P \\ Q \end{cases}$$

then by using the following logical law

$$P \to (Q \to (P \to Q))$$

and applying R.O.D. twice, you get $(P \to Q)$.

**Shortcut Law**: If you have the following two statements:

$$\begin{cases} P \to Q \\ Q \to R \end{cases}$$

then by using the following logical law

$$(P \to Q) \to [(Q \to R) \to (P \to R)]$$

and applying R.O.D. twice, you get $(P \to R)$.

**Law of Conditional/Or**: If you have the following two statements:

$$\begin{cases} P \vee Q \\ Q \to R \end{cases}$$

then by rewriting the first as $\sim P \to Q$, you get the following

$$\begin{cases} \sim P \to Q \\ Q \to R. \end{cases}$$

By applying the Shortcut Law, you get $\sim P \to R$. Therefore, we can conclude that you also get $P \vee R$.

**mR.O.D.**: If you have the following two statements:

$$\begin{cases} P \\ P \leftrightarrow Q \end{cases}$$

then by the Law of And (joining together) and the R.O.D. on the logical law

$$(P \wedge (P \leftrightarrow Q)) \to Q$$

we simply get $Q$. This is similar to R.O.D. but with a biconditional instead of a conditional.

**imR.O.D.**: If you have the following two statements:

$$\begin{cases} Q \\ P \leftrightarrow Q \end{cases}$$

then by the Law of And (joining together) and the R.O.D. on the logical law

$$((P \leftrightarrow Q) \wedge Q) \to P$$

we simply get $P$. This is similar to mR.O.D. but deals with know the right side of the biconditional.

# Survey Questions

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Please take the time to answer the following questions (and answer them seriously). You will not be counted off for not answering the questions, nor will your answer in any way affect your grade, so be honest!

1. What advice would you give a student taking this course next year under the assumption that the same book will be used?

2. What did you like most about this class?

3. What did you like least, besides the book or the amount of material covered, about this class?

4. What did you expect to get out of this class before the semester began (and I do not mean grade-wise here)?

5. What do you feel is the most important concept/idea that you have learned from this course?

6. Do you feel that the instructor could have been more helpful outside of class somehow? If so, how?

7. What do you expect to receive as a final grade, and what do you hope to get?