Math 2283 - Introduction to Logic

Homework #9 - Due 2010.10.27
Proof of Theorem I

Theorem I: $K \cup K \subset K$

Proof: First, we take an instance of Axiom III where $L = M = K$:

$$K \cup K \subset K \leftrightarrow (K \subset K) \wedge (K \subset K) \tag{1}$$

Next, we use an instance of $p \vee p \leftrightarrow p$ with $p = K \subset K$ to get the statement:

$$(K \subset K) \wedge (K \subset K) \leftrightarrow K \subset K \tag{2}$$

Therefore, using the above biconditional (2), and the instance of Axiom III (1), we have

$$K \cup K \subset K \leftrightarrow K \subset K \tag{3}$$

Now, one may be tempted to just say: Therefore $K \cup K \subset K$ by (3), however we must show this logically. To do this, we use the following tautology:

$$(p \leftrightarrow q) \rightarrow (q \rightarrow p) \tag{4}$$

The instance of (4) that we will use is $p = K \cup K \subset K$ and $q = K \subset K$:

$$\{(K \cup K \subset K) \leftrightarrow (K \subset K)\} \rightarrow \{(K \subset K) \rightarrow (K \cup K \subset K)\} \tag{5}$$

Using the Rule of Detachment on (5) and (3) we get

$$(K \subset K) \rightarrow (K \cup K \subset K) \tag{6}$$

Next, notice that the antecedent of (6) is Axiom I. Therefore, we can use the Rule of Detachment on (6) with Axiom I to get

$$K \cup K \subset K \tag{7}$$

And this is Theorem I, proved in ALL its gory detail.