Math 3283 - Foundations

Midterm - 2012.02.23

Solutions

1. Determine which of the following are statements:

   (a) Today is Monday.

   Statement.

   (b) x^2 is positive.

   Not a statement, missing a quantifier and domain of discourse for x.

   (c) $\exists x \in \mathbb{R}\, x^2 > 0$

   Statement, x is quantified.

   (d) $\exists x\, \forall y \in \mathbb{R}\, x^2 \leq y$

   Not a statement, x is missing a domain of discourse.

2. Construct a truth table for the statement $(p \wedge q) \to (\sim q \vee r)$.

p	q	r	$p \wedge q$	$\sim q$	$\sim q \vee r$	$(p \wedge q) \to (\sim q \vee r)$
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

3. The following four statements can be paired into two equivalencies. Determine the equivalencies.

   (a) $(p_1 \vee p_2) \to q$ (b) $(p_1 \to q) \wedge (p_2 \to q)$ (c) $(p_1 \wedge p_2) \to q$ (d) $(p_1 \to q) \vee (p_2 \to q)$

p_1	p_2	q	$(p_1 \vee p_2) \to q$	$(p_1 \to q) \wedge (p_2 \to q)$	$(p_1 \wedge p_2) \to q$	$(p_1 \to q) \vee (p_2 \to q)$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

   Statements (a) and (b) are logically equivalent, and so are (c) and (d).

4. Determine if the following argument is valid.

$p \to (q \vee r)$
$p \vee r$
$\underline{\sim r \qquad\qquad}$
$\therefore\ q$

The arguments is valid. We could perform a truth table analysis, or we can use some reductions. We will go the reduction route. We will number the assumptions (1), (2) and (3) from top to bottom, respectively.

Step 1: (1) $\Leftrightarrow$ $(\sim r \to p)$ (4)
Step 2: (4) $\wedge$ (3) by law of reduction gives: p (5)
Step 3: (5) $\wedge$ (1) by law of reduction gives: $(q \vee r)$ (6)
Step 4: (6) $\Leftrightarrow$ $(\sim r \to q)$ (7)
Step 5: (3) $\wedge$ (7) by law of reduction gives: q (8)
We have now arrived at the desired conclusion. Thus the argument is valid.

5. Determine if each of the following statements are true or false.

(a) $\exists x \in \mathbb{C}\,(4x^8 - 12x^6 + 7x^5 + 2x^3 - 4x + 3 = 0)$

The statement is true since you can factor all polynomials completely over the set of complex numbers $\mathbb{C}$.

(b) $\exists x \in \mathbb{W}\,\forall y, z \in \mathbb{W}\,(x \leq y \cdot z)$

The statement is true, $x = 0$ works.

(c) $\exists! x \in \mathbb{Z}\,\forall y \in \mathbb{Z}\,(x \mid y)$

The statement is true, $x = 1$ works.

6. Negate the following statement: $\forall x\,\exists y\,(p(x,y) \vee \sim q(x))$

$$\sim(\forall x\,\exists y\,(p(x,y) \vee \sim q(x))) \Leftrightarrow \exists x\,\forall y\,(\sim p(x,y) \wedge q(x))$$

7. Prove or disprove: Let n and m be integers, then $n \mid 2m \to n \mid m$.

Disprove: As a counterexample, let $n = 2$ and m be any odd integer.

8. Prove or disprove: let n and m be integers, then nm odd $\leftrightarrow$ n odd and m odd.

This is an iff statement, so we must prove both directions.
$\Leftarrow$: Let us assume that both n and m are odd. Thus, $\exists k, l \in \mathbb{Z}$ such that $n = 2k + 1$ and $m = 2l + 1$. Thus, $nm = (2k+1)(2l+1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$. Since $k, l \in \mathbb{Z}$, $2kl + k + l = r \in \mathbb{Z}$. Therefore $nm = 2r + 1$ which is the definition of nm being odd.

$\Rightarrow$: We prove this direction by contrapositive. Thus, we must prove the statement:

Let n and m be integers, if n even or m even, then nm is even.

So we assume the hypothesis: n even or m even. This is a proof by cases, so we assume n even first. Then by definition, $n = 2k$, for some $k \in \mathbb{Z}$. And so $nm = 2km$. But $k, m \in \mathbb{Z}$, so $km = l \in \mathbb{Z}$ and therefore $nm = 2l$ and we can conclude nm is even. Next, we assume that m is even. By definition, $m = 2k'$, for some $k' \in \mathbb{Z}$. This implies that $nm = 2k'n$, but since $k', n \in \mathbb{Z}$, $k'n = l' \in \mathbb{Z}$ and thus $nm = 2l'$ and we can conclude nm is even.

9. Prove or disprove: Let n and m be integers, then nm even $\leftrightarrow$ n even and m even.

This is false. As an counterexample, let $n = 2$ and $m = 5$. Then $nm = 10$ which is even, but n and m are not both even.

10. Prove by induction that for any positive integer n:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \to q] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_n \to q)]$$

We start the inductive process by showing true for the base case $n = 1$: $(p_1 \to q) \Leftrightarrow (p_1 \to q)$ is trivially true. So now we assume that for for all k such that $1 \leq k \leq n$:

$$[(p_1 \vee p_2 \vee \cdots \vee p_k) \to q] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_k \to q)]$$

We wish to show that for $n + 1$:

$$[(p_1 \vee p_2 \vee \cdots \vee p_{n+1}) \to q] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_{n+1} \to q)]$$

So let us look at the case $k = n$:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \to q] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_n \to q)]$$

If we substitute $p_n = p_n \vee p_{n+1}$, then we have

$$[(p_1 \vee p_2 \vee \cdots \vee (p_n \vee p_{n+1})) \to q] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots ((p_n \vee p_{n+1}) \to q)]$$

However, in the right side's hypothesis we can regroup disjunctions in any way we wish, so we can remove the parenthesis, which gives the right side to be

$$[(p_1 \vee p_2 \vee \cdots \vee p_n \vee p_{n+1}) \to q]$$

The left side, we simple use the fact that for $n = 2$, $((p_n \vee p_{n+1}) \to q) \Leftrightarrow (p_n \to q) \wedge (p_{n+1} \to q)$. Since this is an equivalency, we can substitute into the right side to get

$$[(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots ((p_n \vee p_{n+1}) \to q)] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots ((p_n \to q) \wedge (p_{n+1} \to q)]$$

Since we can conjunction in any order we wish, we now get rid of the parenthesis in the right hand side of the above expression. Thus, we arrive at

$$[(p_1 \vee p_2 \vee \cdots \vee p_{n+1}) \to q] \Leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_{n+1} \to q)]$$