Math 4133 - Linear Algebra

Final Exam - 2011.04.25

Due Date - 2011.05.10 at 10:00 AM

Name: ____________________________

Instructions:

Please show all of your work, allow plenty of space between each problem, or part of problem. You do not have to start each problem on a separate sheet of paper. The only ways that you can get help on any of these problems is to look through books, look through notes, look at old course material on my website, or talk to me. Nothing else is allowed.

The only problems that you are allowed to use *Maple* on are those marked with the symbol [$\mathfrak{M}$]. For these problems, you must print out your results and put them in their proper place in your final exam. I will not accept any electronic files, nor will I accept any *Mathematica* notebook printouts.

Problem Point Distribution:

1.	15 pts	8.	5 pts	13. (c)	5 pts	16. (b)	10 pts
2.	10 pts	9.	10 pts	13. (d)	5 pts	16. (c)	5 pts
3.	15 pts	10. (a)	15 pts	14. (a)	5 pts	16. (d)	5 pts
4.	10 pts	10. (b)	5 pts	14. (b)	5 pts	16. (e)	5 pts
5. (a)	10 pts	11.	5 pts	14. (c)	10 pts		
5. (b)	5 pts	12. (a)	5 pts	15. (a)	5 pts		
6. (a)	10 pts	12. (b)	5 pts	15. (b)	5 pts		
6. (b)	10 pts	12. (c)	15 pts	15. (c)	5 pts		
7. (a)	10 pts	13. (a)	10 pts	15. (d)	5 pts		
7. (b)	5 pts	13. (b)	5 pts	16. (a)	10 pts		

1. Find all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ which transform the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ to parallelograms whose area is 4.

2. Find the set of points (x, y) in $\mathbb{R}^2$ such that the triangle with vertices at $(1,2)$, $(2,4)$ and (x,y) has an area of 10.

3. Solve the following system of equations by LU factorization:

$$\begin{aligned} 3v + 8w + 3x + 5y - 4z &= -2 \\ v - 3w + 5x + 2y - z &= 1 \\ 3v + 2w + x + 5y - 4z &= 8 \\ -2v + 12w - 5x + 7y + 3z &= 3 \end{aligned}$$

4. **[$\mathfrak{M}$]** Use left multiplication by elementary matrices to solve the following system of equations:

$$\begin{aligned} 3i\,w - 4\,x + 5i\,y - 3z &= 2 - 3i \\ (-1+2i)\,w - (2+4i)\,x - 3\,y - (-4+3i)\,z &= 2 \\ 3\,w + i\,x + (-7+2i)\,y + (-6+4i)\,z &= 4 + 2i \\ (2-3i)\,w + (6-7i)\,x + (4-15i)\,y + (4-3i)\,z &= 4 + 5i \end{aligned}$$

5. Consider the following parametric curve:

$$c(t) = (\sin(2\pi t), \sin(\pi t)\cos(2\pi t))$$

(a) Rotate the parametric curve $c(t)$ through an angle of $\theta = \frac{3}{7}\pi$ about the point $P(2, -4)$. Give an explicit formula for the rotated curve.

(b) **[$\mathfrak{M}$]** Graph $c(t)$, the rotated curve, and the point P together.

6. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 2 & 1 \\ 5 & 3 & -1 & 2 \\ 3 & -2 & 3 & 1 \end{bmatrix}$$

(a) Convert A to upper triangular form through left-multiplication by type III elementary matrices.

(b) Compute the determinant of this upper triangular matrix, by hand, and compare the result to that of the determinant of the original matrix A.

7. Consider the basis of $\mathbb{R}^4$ given by

$$\mathbf{B} = \{\langle 1,3,5,3\rangle, \langle 2,4,3,-2\rangle, \langle -1,2,-1,3\rangle, \langle 3,1,2,1\rangle\}$$

(a) Perform the Gram-Schmidt orthonormalization process on this basis $\mathbf{B}$.

(b) Construct a matrix whose columns are the vectors of the orthonormal basis from part (a) and than compute its determinant.

8. Compute the angle between the vectors $\langle 2,-3,0,4,1\rangle$ and $\langle 1,0,-1,3,5\rangle$.

9. If $\mathbb{S} = \operatorname{span}\{\langle 1,-1,1,2,0,1\rangle, \langle 2,2,0,0,1,0\rangle\}$, find $\mathbb{S}^{\perp}$.

10. **[$\mathfrak{M}$]** Consider the following dataset:

$$\big\{(-1.98,-2.13),(-1.45,-3.65),(-1.11,-4.24),(-1.03,0.23),(-0.05,0.92),(0.01,-4.67),(1.03,0.92),$$
$$(0.97,-5.02),(2.1,0.87),(1.97,-4.78),(3.01,-4.24),(3.1,0.3),(3.5,-0.38),(3.98,-2.02)\big\}$$

(a) Find the circle of best fit for the above data. Do not assume the center is at the origin.

(b) Plot these data and the circle of best fit together.

11. Compute the distance from the point $P(-1,2,1,1,3)$ to the plane $2v+3w-4x-y+5z=2$.

12. **[$\mathfrak{M}$]** Consider the following encoded message, which is broken up into six lines of 64 characters (there is no space at the end of each line or at the beginning of the next):

```
JLGLOH.:RCNJT:D.DQVZAOS QTZAWG  MLWLVGLXXWQXBLZAYPARQJTDQJE,ZKMB
-QXCXNUK.:R,GHXOFZCQBGFNOLV:T.:RX:DXJJHK, RAFI EGHKRYLOHS QAIE.A
EVONB KVDURYEBSTNUKQHWLCJ.:RMW-SDA-MTQXB!VQ.,MKTYYWG-A:R IPOAJ:M
KZAFI T.YBYEIIWM:HAQWNBIOS-X,UOF.QXBAEYRQJ:UMVLY.QKWPWRDYZWHBAY-
-QMJ XXWAUQ.IYUGFNYFEOZFDUWGMSPQ.AEEGVTYM,E:JWLU. MJ LNEWM.ZNQTP
IVXIUGIMRYNTMQXBPFQYDVPU:V:T.:RBZKZWHGCBUBAPVUYEIX,UB.VGONWYSVYE
```

(a) Convert the encoded message given above to a matrix of dimension 3×128.

(b) There are 31 characters in this encoded message, with numerical cypher given by:

0 = “A”	1 = “B”	2 = “C”	3 = “D”	4 = “E”	5 = “F”	6 = “G”	7 = “H”
8 = “I”	9 = “J”	10 = “K”	11 = “L”	12 = “M”	13 = “N”	14 = “O”	15 = “P”
16 = “Q”	17 = “R”	18 = “S”	19 = “T”	20 = “U”	21 = “V”	22 = “W”	23 = “X”
24 = “Y”	25 = “Z”	26 = “.”	27 = “,”	28 = “ ”	29 = “-”	30 = “:”	31 = ”!”

Convert the matrix from part (a) using this numerical cypher.

(c) Unfortunately, there was an error in receiving the inverse cypher matrix, which would have allowed you to convert the matrix from part (b) back into the original matrix. You are missing one entry in the matrix. The transmission was cut off before you were able to get the last entry in the matrix, given below:

$$E^{-1} = \begin{bmatrix} 31 & 7 & 11 \\ 0 & 28 & 15 \\ 28 & 19 & \end{bmatrix}$$

Your job is to decipher the message. It is a matter of life and death. Explain how you can decipher this message, then do so!

13. Consider the following differential equation, given in the form $\overrightarrow{x}' = A\overrightarrow{x}$, for $\overrightarrow{x} \in \mathbb{R}^3$:

$$\overrightarrow{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix} \overrightarrow{x}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Compute $e^{A\,t}$.

(c) Solve the ODE from this problem with the initial condition $\overrightarrow{x}(0) = \langle 1, 1, -1 \rangle$.

(d) **[$\mathfrak{M}$]** Plot the solution to this ODE, along with the initial condition.

14. Consider the following linear map T:

$$T(\langle u, v, w, x, y, z \rangle) = \langle u + 4v - 3w, 2v - 6w + 4x, 5w + 2y - 3z, w + y - 4z \rangle$$

(a) Find the matrix A which represents T in terms of matrix multiplication.

(b) Compute the rank of A.

(c) Express both $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ as subspaces of $\mathbb{R}^6$ and $\mathbb{R}^4$, respectively.

15. (a) Construct a system of three planes $\{P_1, P_2, P_3\}$ whose simultaneous solutions is the empty set, yet such that the intersection of any two planes is of dimension 1. Verify this algebraically.

(b) **[$\mathfrak{M}$]** Graph the three planes together.

(c) Construct a system of three planes $\{P_1, P_2, P_3\}$ and a line L whose simultaneous solutions is the empty set, yet such that the intersection of any two planes is of dimension 1 and the intersection of and plane with the line L is empty. Verify this algebraically.

(d) **[$\mathfrak{M}$]** Graph the three planes and the line together.

16. Consider the vector space of all continuous functions on the interval $[-1, 1]$, denoted $\mathrm{C}_{[-1,1]}$, along with the inner product defined for two functions $f(x)$ and $g(x)$ in this vector space by

$$f(x) \cdot g(x) = \int_{-1}^{1} f(x)\, g(x)\, \frac{1}{\sqrt{1 - x^2}}\, dx$$

Notice that this is not just the integral of a product, this is a *weighted integral.* Next, consider the sequence of functions $\{T_n(x)\}_{n=0}^{\infty}$, where $T_0(x) = 1$, $T_1(x) = x$ and in general, $T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$. It can be shown that the set $\{T_n(x)\}_{n=0}^{\infty}$ forms a basis for $\mathrm{C}_{[-1,1]}$.

(a) Using induction, show that $T_n(\cos(\theta)) = \cos(n\theta)$.

(b) Prove the following orthogonality condition:

$$\int_{-1}^{1} T_n(x)\, T_m(x)\, \frac{1}{\sqrt{1 - x^2}}\, dx = \begin{cases} 0 & : n \neq m \\ \pi & : n = m = 0 \\ \frac{\pi}{2} & : n = m \neq 0 \end{cases}$$

(c) **[$\mathfrak{M}$]** Graph $\{T_0(x), T_2(x), \ldots, T_5(x)\}$ together over the interval $[-1, 1]$.

(d) Project the function $F(x) = e^x$ onto the subspace of $\mathrm{C}_{[-1,1]}$ spanned by the set of functions $\{T_0(x), T_2(x), \ldots, T_5(x)\}$.

(e) **[$\mathfrak{M}$]** Graph $F(x) = e^x$ and your answer to part (d) together.