Math 4213 - Complex Analysis

Midterm - 2012.02.27

Solutions

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1. Express the $(2-3i)\overline{(3+4i)}$ in standard form.

$$\begin{aligned}(2-3i)\overline{(3+4i)} &= (2-3i)(3-4i)\\ &= 6-8i-9i+12i^2\\ &= -6-17i\end{aligned}$$

2. Sketch the region D defined by $D = \{z \mid \mathrm{Re}(z) < 0 \text{ and } |z-1+i| < 9\}$

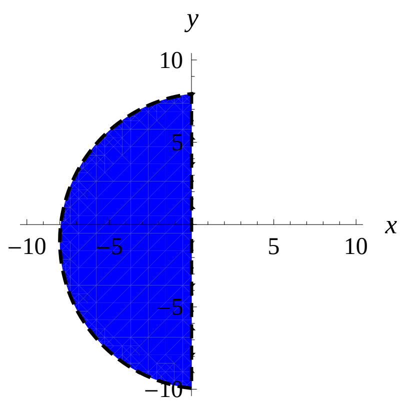

FIGURE 1. Graph depicting D.

3. Compute $\mathrm{Arg}(z)$ if $z = (1-i)^{15}$.

First we need to compute z^{15}:

$$\begin{aligned}(1-i)^{15} &= |1-i|^{15}\left(e^{-i\frac{\pi}{4}}\right)^{15}\\ &= \sqrt{2}^{15}\, e^{-i\,\frac{15}{4}\pi}\end{aligned}$$

Remember that $-\pi < \mathrm{Arg}(z) \leq \pi$, so we need to rewrite the argument $-\frac{15}{4}\pi$ on the correct interval. So we have $\mathrm{Arg}(z) = \frac{\pi}{4}$.

4. Express the function $f(z) = \overline{z}\, e^z$ as $f(x+i\,y) = u(x,y) + i\,v(x,y)$.

First we express $= x + i\,y$ and evaluate $f(x+i\,y)$:

$$\begin{aligned}f(x+i\,y) &= (x-i\,y)e^{x+i\,y}\\ &= (x-i\,y)e^x\, e^{i\,y}\\ &= (x-i\,y)e^x\ (\cos(y)+i\,\sin(y))\\ &= (xe^x - i\,ye^x)\ (\cos(y)+i\,\sin(y))\\ &= xe^x\cos(y) + ye^x\sin(y) + i\,(xe^x\sin(y) - ye^x\cos(y))\end{aligned}$$

So we have $u(x,y) = xe^x\cos(y) + ye^x\sin(y)$ and $v(x,y) = xe^x\sin(y) - ye^x\cos(y)$.

5. Find all values ω such that $\omega^{10} = 2 - 2\,i$.

We are looking for the 10 roots of the complex number $z = 2 - 2\,i$. We rewrite z in $re^{i\theta}$ form as

$$z = 2^{3/2}\, e^{i\,(\frac{7}{4}\pi + 2\pi k)}$$

We therefore have $\omega = z^{1/10}$:

$$\omega_k = 2^{3/20}\, e^{i\,(\frac{7}{40}\pi + \frac{1}{5}\pi k)} \text{ for } k = 0, 1, 2, \ldots, 9$$

6. Consider the function $v(x, y) = 4xy - 3x - 1$.

(a) Verify that $v(x, y)$ is harmonic on $\mathbb{C}$

To verify that $v(x, y)$ is harmonic on $\mathbb{C}$, we compute $v_{xx} + v_{yy}$:

$$v_{xx} + v_{yy} = 0 + 0 = 0$$

Since this holds for all $z \in \mathbb{C}$, and the partial derivatives are continuous everywhere, we have proven that $v(x, y)$ is harmonic on $\mathbb{C}$.

(b) Find the harmonic conjugate $u(x, y)$ to $v(x, y)$

First, we remember that $u_x = v_y$, so

$$\begin{aligned} u(x, y) &= \int v_y(x, y)\, dx + G(y) \\ &= \int 4x\, dx + G(y) \\ &= 2x^2 + G(y) \end{aligned}$$

To determine the value of $G(y)$, we use $u_y = -v_x$:

$$G'(y) = -4y + 3$$

This gives $G(y) = -2y^2 + 3y$. Therefore, we have $u(x, y) = 2x^2 - 2y^2 + 3y$.

(c) Find a function $f(z)$ such that $f(z) = u(x, y) + i\, v(x, y)$. Write your answer in terms of z only, and simplify.

So we have $f(z) = 2x^2 - 2y^2 + 3y + i\,(4xy - 3x - 1)$. We can rewrite this as

$$\begin{aligned} f(z) &= 2x^2 + i\,4xy - 2y^2 - 3i\,(x + i\,y) - i \\ &= 2z^2 - 3i\,z - i \end{aligned}$$

7. Find all *real* numbers c such that the following limit exists. Please express your answer in interval notation.

$$\lim_{n\to\infty} \left(\frac{2}{c} - i\,\frac{3}{2c}\right)^n$$

This is a geometric series, with $z = \frac{2}{c} - i\,\frac{3}{2c}$. So we simply require that $|z| < 1$, which in this case is

$$\begin{aligned} \left|\frac{2}{c} - i\,\frac{3}{2c}\right| &= \left|\frac{1}{c}\right| \left|2 - i\,\frac{3}{2}\right| \\ &= \frac{1}{|c|}\sqrt{4 + \frac{9}{4}} \\ &= \frac{1}{|c|}\sqrt{\frac{25}{4}} \\ &= \frac{1}{|c|}\,\frac{5}{2} < 1 \end{aligned}$$

Solving for c gives

$$|c| > \frac{5}{2}$$

So in interval notation, this is $(-\infty, -\frac{5}{2}) \cup (\frac{5}{2}, \infty)$.

8. Find the set of points z for which the following power series is convergent:

$$g(z) = \sum_{n=2}^{\infty} \frac{(n-i)(n+1)}{(n-1)(2n-i)}(z-2+i)^n$$

Since $c_n = \frac{(n-i)(n+1)}{(n-1)(2n-i)}$, we can attempt to compute ρ by any means, but the ratio test might be the best. So here we go

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1-i)(n+2)}{(n)(2n+2-i)}\frac{(n-1)(2n-i)}{(n-i)(n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{2n^4+O(n^3)}{2n^4+O(n^3)}\right| \\ &= 1\end{aligned}$$

So $\rho = \frac{1}{1} = 1$. Thus, the set of points for which the series converges is $D_1(2-i)$.

9. Find all values of z such that $e^z = 1 - i$.

First we rewrite $1 - i = \sqrt{2}e^{i(-\frac{\pi}{4}+2\pi\, k)}$. We can thus conclude that if $z = x + i\,y$, then $x = \ln(\sqrt{2})$ and $y = -\frac{\pi}{4} + 2\pi\, k$ for k and integer.

10. Find all values of z such that $\text{Log}(z) = 1 - i\frac{\pi}{2}$.

Remember that $\text{Log}(z) = \ln(|z|) + i\,\text{Arg}(z)$. For this problem, we have that $|z| = e$ and $\text{Arg}(z) = -\frac{\pi}{2}$. Therefore, we have $z = re^{i\theta} = e\,e^{-i\frac{\pi}{2}} = -i\,e$.

11. Evaluate the integral $\int_{\mathcal{C}} z^3\cos(z) + z^2\sin(z)\,dz$, where $\mathcal{C}$ is the upper 1/2 of the circle of radius 7, centered at $z_0 = 0$ and oriented positively.

First, we note that the function $f(z) = z^3\cos(z) + z^2\sin(z)$ is entire on $\mathbb{C}$. And if we consider the contour $\mathcal{C}'$ to be the contour along the x axis from -7 to 7. Thus, we have

$$\int_{\mathcal{C}+\mathcal{C}'} z^3\cos(z) + z^2\sin(z)\,dz = 0$$

since the function is entire on the simply connected domain corresponding to the interior of $\mathcal{C} + \mathcal{C}'$. Furthermore, since $f(x) = x^3\cos(x) + x^2\sin(x)$ is odd, we have the following:

$$\begin{aligned}\int_{\mathcal{C}'} z^3\cos(z) + z^2\sin(z)\,dz &= \int_{-7}^{7} x^3\cos(x) + x^2\sin(x)\,dx \\ &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= \int_{\mathcal{C}+\mathcal{C}'} z^3\cos(z) + z^2\sin(z)\,dz \\ &= \int_{\mathcal{C}} z^3\cos(z) + z^2\sin(z)\,dz + \int_{\mathcal{C}'} z^3\cos(z) + z^2\sin(z)\,dz \\ &= \int_{\mathcal{C}} z^3\cos(z) + z^2\sin(z)\,dz + 0 \\ &= \int_{\mathcal{C}} z^3\cos(z) + z^2\sin(z)\,dz\end{aligned}$$

and thus, our integral is 0.

Note, also that since we have a nice analytic function on all of $\mathbb{C}$, we can simply transform the contour $\mathcal{C}$ to the contour $\mathcal{C}'$, which would result in a value of zero as well.

12. Consider the function $f(z) = \frac{1}{z^4-1}$. Compute $\displaystyle\int_{\mathcal{C}} f(z)\,dz$ where $\mathcal{C}$ is the positively oriented ellipse centered at $z_0 = i/2$ and semi-major axis of length 2 in the x-direction and semi-minor axis of length 1 in the y-direction.

First, we rewrite $f(z)$ using partial fractions:

$$f(z) = \frac{1}{4}\frac{1}{z-1} - \frac{1}{4}\frac{1}{z+1} + \frac{i}{4}\frac{1}{z-i} - \frac{i}{4}\frac{1}{z+i}$$

The contour $\mathcal{C}$ encloses the points $z = i$ and $z = \pm 1$, but not the point $z = -i$. Therefore, by the *Extended Cauchy-Goursat Theorem*, we simply compute the integral of $f(z)$ around contours of small enough radii so that they do not intersect. We therefore end up with

$$\begin{aligned}\int_{\mathcal{C}} f(z)\,dz &= \frac{1}{4}2\pi i - \frac{1}{4}2\pi i + \frac{i}{4}2\pi i - \frac{i}{4}0 \\ &= -\frac{\pi}{2}\end{aligned}$$