Math 4213 - Complex Analysis

Quiz #14 - 2012.03.05

Solutions

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1. On what type of region is a Laurent series convergent?

   The region of convergence for a Laurent series is an annulus.

2. Compute the following integral:
$$\int_{\mathcal{C}} \frac{\sin(z)}{(z-i)^2(z+2)^2}\,dz$$
where $\mathcal{C}$ is a rectangular contour, oriented counter clockwise, with corner points $\{-1+2i, 1+2i, 1-i, -1-i\}$.

   Since only the point $z=i$ lies inside the contour, by Cauchy-Goursat, we can rewrite this as
$$\int_{\mathcal{C}_r^+(i)} \frac{\sin(z)}{(z+2)^2}\,\frac{1}{(z-i)^2}\,dz$$
where $0 < r < \sqrt{5}$ is any value so that $\mathcal{C}_r^+(i)$ does not contain the point $z=-2$. By Cauchy's Integral Formula, we have that
$$\int_{\mathcal{C}_r^+(i)} \frac{\sin(z)}{(z+2)^2}\,\frac{1}{(z-i)^2}\,dz = \frac{2\pi i}{1!} f'(i)$$
where $f(z) = \frac{\sin(z)}{(z+2)^2}$. So we simply need to compute the derivative of $f(z)$, which is
$$f'(z) = \frac{\cos(z)(z+2) - 2\sin(z)}{(z+2)^3}$$
and thus
$$\begin{aligned}\int_{\mathcal{C}_r^+(i)} \frac{\sin(z)}{(z+2)^2}\,\frac{1}{(z-i)^2}\,dz &= \frac{2\pi i}{1!}\,\frac{\cos(i)(i+2) - 2\sin(i)}{(i+2)^3} \\ &= \frac{2\pi i}{1!}\,\frac{\cosh(1)(i+2) + 2\sinh(1)}{(i+2)^3}\end{aligned}$$