Math 4213 - Complex Analysis

Quiz #15 - 2012.03.09

Solutions

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1. What is the relationship between the residue of a function $f(z)$ at z_0 and it's Laurent series at z_0?

   The residue of $f(z)$ at $z = z_0$ is simply the a_{-1} coefficient in the Laurent series of $f(z)$ at z_0.

2. Compute the following integral:

$$\int_{\mathcal{C}_2^+(0)} \frac{e^z + 1}{(z-3i)^2(z^2+1)}\, dz$$

   First, we rewrite the integrand as

$$\int_{\mathcal{C}_2^+(0)} \frac{e^z + 1}{(z-3i)^2(z^2+1)}\, dz = \int_{\mathcal{C}_2^+(0)} \frac{e^z + 1}{(z-3i)^2(z+i)(z-i)}\, dz$$

The contour $\mathcal{C}_2^+(0)$ encloses the simple poles at $z = i$ and $z = -i$, but not the pole of order 2 at $z = 3i$. Thus, by Cauchy-Goursat, we have

$$\int_{\mathcal{C}_2^+(0)} \frac{e^z + 1}{(z-3i)^2(z+i)(z-i)}\, dz = \int_{\mathcal{C}_{r_1}^+(i)} \frac{e^z + 1}{(z-3i)^2(z+i)} \frac{1}{(z-i)}\, dz + \int_{\mathcal{C}_{r_2}^+(-i)} \frac{e^z + 1}{(z-3i)^2(z-i)} \frac{1}{(z+i)}\, dz$$

where r_1 and r_2 are chosen so the contours contain only the poles about which each is centered. We now have

$$\int_{\mathcal{C}_{r_1}^+(i)} \frac{e^z + 1}{(z-3i)^2(z+i)} \frac{1}{(z-i)}\, dz = 2\pi\, i\, \frac{e^i + 1}{(-2i)^2(2i)} = 2\pi\, i\, \frac{e^i + 1}{-8i}$$

and

$$\int_{\mathcal{C}_{r_2}^+(-i)} \frac{e^z + 1}{(z-3i)^2(z-i)} \frac{1}{(z+i)}\, dz = 2\pi\, i\, \frac{e^i + 1}{(-4i)^2(-2i)} = 2\pi\, i\, \frac{e^{-i} + 1}{32i}$$

Adding these up, we have

$$\begin{aligned}\int_{\mathcal{C}_2^+(0)} \frac{e^z + 1}{(z-3i)^2(z+i)(z-i)}\, dz &= 2\pi\, i \left[\frac{e^i + 1}{-8i} + \frac{e^{-i} + 1}{32i}\right] \\ &= \pi \left[-\frac{e^i + 1}{4} + \frac{e^{-i} + 1}{16}\right]\end{aligned}$$