Math 2283 - Introduction to Logic
Final Exam

Assigned: 2008.11.25 at 8:30

Due: 2008.12.10 at 13:00

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire, but you do have to attach the end of exam survey questions. Remember, you have two whole weeks to work on this, and it will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

Printed Name: ____________________________________

Signature: ____________________________________

Date: ____________________________________

1) Consider the sentential function $p\Delta q$ as an abbreviation of the expression: *neither p nor q*. Construct the fundamental truth table for this function, which would comply with the intuitive meaning ascribed to the symbol Δ. Then verify by truth tables that the following sentences are true and thus may be accepted as laws of sentential calculus:

a) $(\sim p) \leftrightarrow (p\Delta p)$

b) $(p \vee q) \leftrightarrow [(p\Delta q)\Delta(p\Delta q)]$

c) $(p \rightarrow q) \leftrightarrow \{[(p\Delta p)\Delta q]\,\Delta\,[(p\Delta p)\Delta q]\}$

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2) Prove the following theorem:

$$p \rightarrow [\ (p \rightarrow q) \rightarrow q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

Theorem I: $[\ p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$

Theorem II: $p \rightarrow p$

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3) Prove the following theorem:

$$\text{if } x \leq y,\ \ y < z \text{ and } z \leq t,\ \text{ then } x < t,$$

using ONLY the following axiom:

$$\text{if } x < y \text{ and } y < z,\ \text{ then } x < z.$$

Here $(x \leq y) \leftrightarrow [(x < y) \vee (x = y)]$ is the standard definition of the $\leq$ symbol. You may assume all knowledge of logic, and hence may employ any shortcuts as needed. Be sure to state what laws/shortcuts you are using at each step in your proof.

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4) Show that between any two real numbers, EXACTLY three of the following six relations hold:

$$=,\ <,\ >,\ \neq,\ \leq,\ \geq$$

And no, you may NOT use the number line to prove this.

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5) Consider the set $\mathbb{S} = \{a, b, c\}$ and let the operation $\diamond$ on the elements of $\mathbb{S}$ be defined as follows:

$$a \diamond a = a, \quad a \diamond b = b, \quad a \diamond c = c$$
$$b \diamond a = b, \quad b \diamond b = c, \quad b \diamond c = a$$
$$c \diamond a = c, \quad c \diamond b = a, \quad c \diamond c = b$$

Determine whether the set $\mathbb{S}$ is an Abelian group with respect to the operation $\diamond$. Be sure to show all your work.

6) If we assume that the following two statements are true:

$$\begin{cases} p_1 \to q_1 \\ p_2 \to q_2 \end{cases}$$

it seems that the following statement should also hold:

$$(p_1 \wedge p_2) \to (q_1 \vee q_2)$$

Construct a proof that this is indeed the case. You MAY NOT use truth tables to show this.

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7) Let $\mathbb{K}$ be the set of all numbers strictly between 0 and 1, i.e. $\mathbb{K} = \{x \mid 0 < x < 1\}$. Let $\mathbb{R}$ denote, as usual, the set of all real numbers. Prove by construction of a biunique function that $\mathbb{K}$ and $\mathbb{R}$ are equinumerous.

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8) Given a class $\mathbb{K}$ and a relation R, consider the following set of axioms (you may assume $x, y, z, t \in \mathbb{K}$):

Axiom 1: $\underset{x}{\mathbf{A}}\ xRx$
Axiom 2: $\underset{x,y,z}{\mathbf{A}}\ (xRz \wedge yRz) \to xRy$
Axiom 3: $\underset{y,z}{\mathbf{A}}\ yRz \to zRy$
Axiom 4: $\underset{x,y,z,t}{\mathbf{A}}\ (xRy \wedge yRz \wedge zRt) \to xRt$

Prove that the axiomatic system consisting of Axiom 1 and Axiom 2 is equipollant to the axiomatic system consisting of Axiom 1, Axiom 3 and Axiom 4.

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9) Prove the following statements regarding relations:
a) If the relation R is transitive and irreflexive in a class, then R is also asymmetric in that class.
b) If the relation R is irreflexive and transitive in a class, then R' is connected in that class.

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10) Consider the class $\mathbb{K}$ of possible sets formed by an arbitrary group of objects, the two relations $\subseteq$, $\supseteq$ and two operations $\cup$ and $\cap$. Are either of the given operations monotonic in the class $\mathbb{K}$ with respect to either of the relations $\subseteq$ and $\supseteq$?

Remember the definitions of the two relations $\subseteq$ and $\supseteq$ are

$$A \subseteq B \leftrightarrow \underset{x}{\mathbf{A}}\ (x \in A \to x \in B) \text{ and } A \supseteq B \leftrightarrow \underset{x}{\mathbf{A}}\ (x \in B \to x \in A)$$

respectively with $A, B \in \mathbb{K}$. Also, do not forget the universal and null sets.

Common Laws of Logic

The following is a list of laws that we have used a decent amount in class or are just silly and may prove useful for this exam:

Law of And (breaking apart): If you have the following statement:

$$P \wedge Q$$

then by using the following logical law

$$(P \wedge Q) \to Q,$$

and applying L.O.D. you get Q.

Law of And (joining together): If you have the following two statements:

$$\begin{cases} P \\ Q \end{cases}$$

then by using the following logical law

$$P \to (Q \to (P \wedge Q))$$

and applying L.O.D. twice, you get $(P \wedge Q)$.

Law of Or (single): If you have the following statement:

$$P$$

then by using the following logical law

$$P \to (P \vee Q)$$

and applying L.O.D. you get $(P \vee Q)$.

Law of Or (double): If you have the following two statements:

$$\begin{cases} P \\ Q \end{cases}$$

then by using the following logical law

$$P \to (Q \to (P \vee Q))$$

and applying L.O.D. twice, you get $(P \vee Q)$.

Law of Or (repeat): If you have the following statement:

$$P \vee P$$

then by using the following logical law

$$(P \vee P) \to P$$

and applying L.O.D., you get just P.

Law of Conditional: If you have the following two statements:

$$\begin{cases} P \\ Q \end{cases}$$

then by using the following logical law

$$P \to (Q \to (P \to Q))$$

and applying L.O.D. twice, you get $(P \to Q)$.

Shortcut Law: If you have the following two statements:

$$\begin{cases} P \to Q \\ Q \to R \end{cases}$$

then by using the following logical law

$$(P \to Q) \to [(Q \to R) \to (P \to R)]$$

and applying L.O.D. twice, you get $(P \to R)$.

Law of Conditional/Or: If you have the following two statements:

$$\begin{cases} P \vee Q \\ Q \to R \end{cases}$$

then by rewriting the first as $\sim P \to Q$, you get the following

$$\begin{cases} \sim P \to Q \\ Q \to R. \end{cases}$$

By applying the Shortcut Law, you get $\sim P \to R$. Therefore, we can conclude that you also get $P \vee R$.

Survey Questions

Please take the time to answer the following questions (and answer them seriously). You will not be counted off for not answering the questions, nor will your answer in any way affect your grade, so be honest!

1) What advice would you give a student taking this course next year under the assumption that the same book will be used?

2) What did you like most about this class?

3) What did you like least, besides the book or the amount of material covered, about this class?

4) What did you expect to get out of this class before the semester began (and I do not mean grade-wise here)?

5) What do you feel is the most important concept/idea that you have learned from this course?

6) Do you feel that the instructor could have been more helpful outside of class somehow? If so, how?

7) What do you expect to receive as a final grade, and what do you hope to get?

Trivia Questions

When you are having a hard time solving some of the problems in this exam, perhaps you can take a breather and see if you know the answer to one of the following questions. If you know the answer to any of these questions without looking them up, I will be totally amazed and think you cool (which will in itself probably make you uncool with respect to your friends). *Hint: Five out of six of these are references to songs, and one is to a TV show.*

1) According to James Herbert, what animal should we let wear glasses?

2) For Al J., every day is what?

3) The glow from Robert D. N.'s TV set glows blue like what?

4) If you said "I think it's dark and it looks like rain", how should I respond in true Robert S. fashion?

5) All the greed is nothing that Chris C. needs if he can't face the what?

6) According to Blaine, playing what for years helps you learn a little something about courage?