Math 2283 - Introduction to Logic

Quiz #8 - 2008.10.29

Solutions

Consider the following axiomatic system where we assume the symbols $\rightarrow$, $\leftrightarrow$ (defined as usual) as primitive terms, and the these five sentences as axioms:

Ax I: $(p \leftrightarrow q) \rightarrow (q \rightarrow p)$

Ax II: $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$

Ax III: $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$

Ax IV: $(p \rightarrow q) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)]$

Ax V: $[p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$

Furthermore, we agree to apply in the proof of the following theorem only the rule of substitution and the law of detachment (L.O.D.).
Prove the following Theorem:

Theorem: $(q \rightarrow p) \rightarrow [(q \leftrightarrow p) \rightarrow (p \leftrightarrow q)]$

Hint 1: Axiom III above is the same one that you have been using for the shortcut method described in class, so you can use the shortcut from class if need be.

Hint 2: You will need to swap p and q in either Ax I or Ax II.

Hint 3: The result of Hint 2 and Ax IV should be compared.

Hint 4: The result of Hint 3 should be compared to the theorem that you want to prove, then use an axiom that you have not yet utilized. You will need to do a serious substitution in this axiom.

Hint 5: The L.O.D. will finish the proof of the theorem.

First, we swap p and q in Ax I to get

(1) $$(q \leftrightarrow p) \rightarrow (p \rightarrow q)$$

Notice that the antecedent of Ax IV is the consequent of (1):

$$\begin{cases} (q \leftrightarrow p) \rightarrow (p \rightarrow q) \\ (p \rightarrow q) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)]\,, \end{cases}$$

so we use Ax III and the shortcut method on the above two sentences to get:

(2) $$(q \leftrightarrow p) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)]$$

This is almost what we want to prove, however the $(q \leftrightarrow p)$ term and $(q \rightarrow p)$ are switched. This is where Ax V comes in. if we let $p = q \leftrightarrow p$, $q = q \rightarrow p$ and $r = p \leftrightarrow q$ we get

(3) $$[(q \leftrightarrow p) \rightarrow ((q \rightarrow p) \rightarrow (p \leftrightarrow q))] \rightarrow [(q \rightarrow p) \rightarrow ((q \leftrightarrow p) \rightarrow (p \leftrightarrow q))]$$

The antecedent of (3) is (2), i.e. we have

$$\begin{cases} (q \leftrightarrow p) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)] \\ [(q \leftrightarrow p) \rightarrow ((q \rightarrow p) \rightarrow (p \leftrightarrow q))] \rightarrow [(q \rightarrow p) \rightarrow ((q \leftrightarrow p) \rightarrow (p \leftrightarrow q))]\,. \end{cases}$$

So by the (L.O.D.) we get the consequent of (3):

$$(q \rightarrow p) \rightarrow ((q \leftrightarrow p) \rightarrow (p \leftrightarrow q))$$

This is exactly the theorem that we were trying to prove.