# Math 2283 - Introduction to Logic

**Quiz #8 - 2008.10.29**

**Solutions**

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Consider the following axiomatic system where we assume the symbols $\rightarrow$, $\leftrightarrow$ (defined as usual) as primitive terms, and these five sentences as axioms:

Ax I: $p \rightarrow (q \rightarrow p)$

Ax II: $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$

Ax III: $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$

Ax IV: $[p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$

Furthermore, we agree to apply in the proof of the following theorem only the rule of substitution and the law of detachment (L.O.D.).
Prove the following theorem:

Theorem: $(p \leftrightarrow q) \rightarrow [p \rightarrow (p \rightarrow q)]$

Hint 1: Ax III above is the same one that you have been using for the shortcut method described in class, so you can use the shortcut from class if need be.

Hint 2: Applying Ax III (shortcut) to Ax II and Ax III is a good first step.

Hint 3: Ax IV should be used next with a suitable substitution.

Hint 4: Now find a suitable substitution for $r$ in the result of hint 3.

Hint 5: If done correctly, the antecedent of the result of hint 4 should look like Ax I after swapping variables in Ax I.

First, we use the shortcut on Ax II and Ax III as suggested in the hint. We get:

(1) $$(p \leftrightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$$

Next, we set $p = (p \leftrightarrow q)$, $q = (q \rightarrow r)$ and $r = (p \rightarrow r)$ in Ax IV:

(2) $$[(p \leftrightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))] \rightarrow [(q \rightarrow r) \rightarrow ((p \leftrightarrow q) \rightarrow (p \rightarrow r))]$$

Notice that (1) is the antecedent of (2) due to our substitution, thus using L.O.D. on (1) and (2) we get

(3) $$(q \rightarrow r) \rightarrow ((p \leftrightarrow q) \rightarrow (p \rightarrow r)).$$

Looking back at the theorem we are trying to prove, we note that if $r = p \rightarrow q$ we would be close. So we do this and get

(4) $$(q \rightarrow (p \rightarrow q)) \rightarrow ((p \leftrightarrow q) \rightarrow (p \rightarrow (p \rightarrow q))).$$

The antecedent of (4) is Ax I with $p$ and $q$ swapped, so if we swap $p$ and $q$ in Ax I, together with (4) we have the pair of sentences:

$$\begin{cases} q \rightarrow (p \rightarrow q) \\ (q \rightarrow (p \rightarrow q)) \rightarrow ((p \leftrightarrow q) \rightarrow (p \rightarrow (p \rightarrow q))). \end{cases}$$

Using L.O.D. gives

$$(p \leftrightarrow q) \rightarrow (p \rightarrow (p \rightarrow q)),$$

which was exactly what we were trying to prove.