Math 2315 - Calculus II

Final Exam - 2007.11.26

Due Date - 2007.12.10 - 11:00 A.M.

Name: ____________________________

Instructions

Write each problem on its own page and be sure to show all your work. Compose your answers in a very concise and neat manner.

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1. Compute the following integrals.

a)

$$\int \frac{d\theta}{\cos^4(\theta)}$$

b)

$$\int \left(4 - x + \frac{1}{3}x^2 + 6x^3\right) e^{2-3x} dx$$

c)

$$\int \left(x + 6x^3 - 7x^5\right) e^{-3x^2} dx$$

d)

$$\int \frac{dt}{\cosh^2(t) + \sinh^2(t)}$$

e)

$$\int \frac{\tan^{-1}(u)}{u^2} du$$

f)

$$\int \frac{(x+1)(x+2)(x+3)}{(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)} dx$$

2. Use the Taylor expansion of $f(x) = \ln(1+x)$ about $x = 0$ to find the value of the following convergent series:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k}\right)$$

3. Consider the following sequence:

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln(n)$$

a) Show that $a_n \geq 0$ for $n \geq 1$.

b) Show that $a_{n+1} \leq a_n$ for all n.

c) Prove that $\lim_{n\to\infty} a_n$ exists.

4. If you remember from class, the following sum is divergent:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \left(1 + \frac{1}{k}\right)$$

a) Find a formula for S_{2n} and S_{2n+1}.

b) Compute $\lim_{n\to\infty} S_{2n}$ and $\lim_{n\to\infty} S_{2n+1}$ exactly (do not approximate). Problem 2 might be of some help. This is an example where the partial sums are bounded but do not converge.

5. Calculate the total area enclosed by the curve $r^2 = \cos(\theta)e^{\sin(\theta)}$.

6. Show that if $0 \leq f'(x) \leq 1$ for all x, then the arc length of $y = f(x)$ over $[a, b]$ is at most $\sqrt{2}(b-a)$. Show that for $f(x) = x$, the arc length is equal to $\sqrt{2}(b-a)$.

7. The following equation arises in the description of Bose-Einstein condensation (the quantum theory of gases cooled to near absolute zero):

$$A_0 = \frac{4A}{\sqrt{\pi}} \int_0^\infty \frac{x^2 e^{-x^2}}{1 - Ae^{-x^2}} dx$$

It is necessary to derive an approximate expression for A_0 in terms of A for $|A|$ small.

a) Show that the second Maclaurin polynomial $T_2(A)$ for the function

$$f(A) = \frac{x^2 e^{-x^2}}{1 - Ae^{-x^2}}$$

(where A is the variable and x is treated as a constant) is

$$T_2(A) = x^2 e^{-x^2} + x^2 e^{-2x^2} A + x^2 e^{-3x^2} A^2.$$

b) Use the approximation $A_0 \approx \frac{4A}{\sqrt{\pi}} \int_0^\infty T_2(A) dx$ to show that

$$A_0 \approx A + \frac{1}{2\sqrt{2}} A^2 + \frac{1}{3\sqrt{3}} A^3.$$

You may use the formula (valid for $\lambda > 0$)

$$\int_0^\infty x^2 e^{-\lambda x^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}}.$$

8. Rewrite the following statement and sign your name to it:

"I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one."