Math 2315 - Calculus II

Homework #10 - 2007.11.02

Due Date - 2007.11.09

Solutions

Part 1: Problems from sections 11.1 - 11.2

Part 2: The *fun* problems.

1. If $a_n = \sqrt{n+3} - \sqrt{n}$, compute $\lim_{n\to\infty} a_n$.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{n+3} - \sqrt{n} \\
&= \lim_{n\to\infty} \sqrt{n+3} - \sqrt{n} \cdot \frac{\sqrt{n+3} + \sqrt{n}}{\sqrt{n+3} + \sqrt{n}} \\
&= \lim_{n\to\infty} \frac{n+3-n}{\sqrt{n+3} + \sqrt{n}} \\
&= \lim_{n\to\infty} \frac{3}{\sqrt{n+3} + \sqrt{n}} \\
&= 0.
\end{aligned}$$

2. If $b_n = n^2 \left(\sqrt[3]{n^3+1} - n\right)$, compute $\lim_{n\to\infty} b_n$.

$$\begin{aligned}
\lim_{n\to\infty} b_n &= n^2 \left(\sqrt[3]{n^3+1} - n\right) \\
&= \lim_{n\to\infty} n^2 \sqrt[3]{n^3+1} - n^3 \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}\sqrt[3]{n^3+1} - 1}{\frac{1}{n^3}} \\
&= \lim_{n\to\infty} \frac{\sqrt[3]{1 + \frac{1}{n^3}} - 1}{\frac{1}{n^3}}
\end{aligned}$$

This is now in the form $\frac{0}{0}$ so we apply l'Hospital's rule to get

$$\lim_{n\to\infty} \frac{\sqrt[3]{1+\frac{1}{n^3}}-1}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{3}\left(1+\frac{1}{n^3}\right)^{-\frac{2}{3}}\left(-3n^{-4}\right)}{-3n^{-4}}$$
$$= \frac{1}{3}.$$

3. Let $c_n = \frac{\sqrt[n]{n!}}{n}$

a) Show that

$$\ln(c_n) = \frac{\ln(n!) - n\ln(n)}{n}.$$

We show this as follows:

$$\ln\left(\frac{\sqrt[n]{n!}}{n}\right) = \ln\left((n!)^{\frac{1}{n}}\right) - \ln(n)$$
$$= \frac{1}{n}\ln(n!) - \ln(n)$$
$$= \frac{\ln(n!) - n\ln(n)}{n}.$$

b) Show that

$$\frac{\ln(n!) - n\ln(n)}{n} = \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right).$$

We show this as follows:

$$\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right) = \frac{1}{n}\left(\sum_{k=1}^{n}\ln(k) - \ln(n)\right)$$
$$= \frac{1}{n}\left(\ln(1)+\ln(2)+\cdots+\ln(n) - n\ln(n)\right)$$
$$= \frac{1}{n}\left(\ln(1\cdot 2\cdots n) - n\ln(n)\right)$$
$$= \frac{\ln(n!) - n\ln(n)}{n}.$$

c) Show that $\ln(c_n)$ converges to $\int_0^1 \ln(x)dx$.

Notice that as a Riemann sum with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$, and $x_k = 0 + k\delta x = \frac{k}{n}$. So we have

$$\int_0^1 \ln(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right).$$

Since

$$\ln(c_n) = \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right),$$

we have that

$$\lim_{n\to\infty} \ln(c_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right) = \int_0^1 \ln(x)dx.$$

d) Prove that $\lim_{n\to\infty} c_n = \frac{1}{e}$.

First we compute

$$\begin{aligned}
\int_0^1 \ln(x)dx &= (x\ln(x) - x)\Big|0^1 \\
&= -1 - \lim_{\alpha\ to 0} (\alpha \ln(\alpha) - \alpha) \\
&= -1 - \lim_{\alpha\to 0} \frac{\ln(\alpha)}{\frac{1}{\alpha}} + \lim_{\alpha\to 0} \alpha \\
&= -1 - \lim_{\alpha\to 0} \frac{\frac{1}{\alpha}}{-\frac{1}{\alpha^2}} + 0 \\
&= -1 \lim_{\alpha\to 0} (-\alpha) \\
&= -1.
\end{aligned}$$

Since $\lim_{n\to\infty} c_n = -1$, we can conclude that $\lim_{n\to\infty} c_n = e^{-1}$.

4. Prove that if a is a positive integer, then

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \frac{1}{a}\left(1 + \frac{1}{2} + \cdots + \frac{1}{a}\right).$$

First, by partial fraction we have that

$$\frac{1}{n(n+a)} = \frac{1}{a}\frac{1}{n} - \frac{1}{a}\frac{1}{n+a},$$

so

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+a)} &= \sum_{n=1}^{\infty}\frac{1}{a}\frac{1}{n}-\frac{1}{a}\frac{1}{n+a}\\ &= \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+a}\\ &= \frac{1}{a}\left(1+\frac{1}{2}+\cdots+\frac{1}{a}\right).\end{aligned}$$

How do we get that last step? Well if you notice, the only terms that will not eventually cancel are those terms of the form $\frac{1}{n}$ for $1 \leq n \leq a$, since the subtraction term $\frac{1}{n+a}$ starts at $\frac{1}{a+1}$.

5. A ball dropped from a height of 100 ft begins to bounce. Each time it strikes the ground, it returns to two-thirds of its previous height. What is the total distance traveled by the ball if it bounces infinitely many times?

The first time the ball drops, it travels a distance of 100 ft to the ground. Then it travels $100 \cdot \frac{2}{3}$ up and returns the same distance back down and hits the ground again. It then travels a distance of $100 \cdot \left(\frac{2}{3}\right)^2$ back up and back down again etc... So we have that the distance D traveled is given by

$$\begin{aligned}D &= 100 + 2\cdot 100\cdot\frac{2}{3} + 2\cdot 100\cdot\left(\frac{2}{3}\right)^2 + 2\cdot 100\cdot\left(\frac{2}{3}\right)^3 + \cdots\\ &= 100 + 200\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k\\ &= 100 + 200\left(\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} - 1\right)\\ &= 100 + 200\left(\frac{1}{1-\frac{2}{3}} - 1\right)\\ &= 100 + 200(3-1)\\ &= 500 \text{ ft.}\end{aligned}$$

6. If $\sum_{k=1}^{\infty} a_k$ is convergent, is it necessarily true that adding or removing a finite number of terms to the sum must yield a series which is also convergent? If not,

give an example to prove your point.

Since the finite sum of a finite set of numbers is finite, it must be true. Whether it is added or subtracted from the original series makes no difference.

7. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+a} = \sum_{n=a+1}^{\infty} \frac{1}{n-a} - \frac{1}{n}.$$

We start with the sum

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+a},$$

and simply replace n with $r - a$ everywhere:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+a} &= \sum_{r-a=1}^{\infty} \frac{1}{r-a} - \frac{1}{r-a+a} \\ &= \sum_{r=a+1}^{\infty} \frac{1}{r-a} - \frac{1}{r}. \end{aligned}$$

Replacing the variable r with n in the final line above gives the result.