Math 2315 - Calculus II

Homework #11 - 2007.11.12

Due Date - 2007.11.20

Solutions

Part 1: Problems from sections 11.3 - 11.6

Part 2: The *fun* problems.

For problems 1 through 5, determine whether or not the series converges.

1.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$$

If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = 1.$$

Since $\lim_{n\to\infty} a_n = 1$, the sum does not converge.

2.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\sin(n)}$$

Notice that if $n > 1$, we have

$$\frac{1}{n^2+1} \leq \frac{1}{n^2+\sin(n)} \leq \frac{1}{n^2-1},$$

thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+\sin(n)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2-1}.$$

Since we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

we can use the Limit Comparison Test with $\frac{1}{n^2}$ to show that the sums on involving $\frac{1}{n^2+1}$ and $\frac{1}{n^2-1}$ both converge. Therefore, we know that the original sum converges.

3.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$$

Since

$$\lim_{n\to\infty}\left|(-1)^{n-1}\frac{1}{\sqrt{n}}\right| = \lim_{n\to\infty}\left|\frac{1}{\sqrt{n}}\right| = 0,$$

the alternating series is convergent.

4.

$$\sum_{n=4}^{\infty}(-1)^{n}\tan\left(\frac{1}{n}\right)$$

Since

$$\lim_{n\to\infty}\left|(-1)^{n}\tan\left(\frac{1}{n}\right)\right| = \lim_{n\to\infty}\left|\tan\left(\frac{1}{n}\right)\right| = 0,$$

the alternating series is convergent.

5.

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{1}{n}\right)}{n^2}$$

There are several ways to prove this one. Notice that

$$\lim_{n\to\infty}\cos\left(\frac{1}{n}\right) = 1,$$

and

$$\cos\left(\frac{1}{n}\right) \leq 1, \quad \forall n.$$

Therefore, we get

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{1}{n}\right)}{n^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2},$$

so the sum converges.

6. For what values of x does the following sum converge?

$$\sum_{n=1}^{\infty}2^{n}x^{n}$$

We write this as

$$\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} 2x\,(2x)^{n-1}\,,$$

which is in the form for a geometric series. This requires that

$$|2x| < 1 \to -\frac{1}{2} < x < \frac{1}{2}.$$

7. For what powers k does the following converge?

$$\sum_{n=1}^{\infty} n^k 3^{-n}$$

We rewrite as

$$\sum_{n=1}^{\infty} n^k 3^{-n} = \sum_{n=1}^{\infty} \frac{n^k}{3^n}$$

If we perform the Ratio Test, we have that

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^k}{3^{n+1}}}{\frac{n^k}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^k}{n^k} \frac{3^n}{3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^k}{n^k} \frac{1}{3} \right| = \frac{1}{3}.$$

Since this was independent of k, the sum converges for all k.