Math 2315 - Calculus II

Homework #2 - 2007.08.31

Due Date - 2007.09.07

Solutions

Part 1: Problems from sections 7.3 and 7.4.
Section 7.3:

19.

$$\int \frac{\sqrt{1+x^2}}{x}dx$$

We will use the substitution $x = \cot(\theta)$, $dx = -\csc^2(\theta)$. This gives

$$\int \frac{\sqrt{1+x^2}}{x}dx = -\int \frac{\sqrt{1+\cot^2(\theta)}}{\cot(\theta)}\csc^2(\theta)d\theta.$$

Next, some algebraic manipulations:

$$\begin{aligned}
-\int \frac{\sqrt{1+\cot^2(\theta)}}{\cot(\theta)}\csc^2(\theta)d\theta &= -\int \tan(\theta)\csc^3(\theta)d\theta \\
&= -\int \csc^2(\theta)\csc(\theta)\tan(\theta)d\theta \\
&= -\int \left(1+\cot^2(\theta)\right)\csc(\theta)\tan(\theta)d\theta \\
&= -\int \csc(\theta)\tan(\theta) + \csc(\theta)\cot(\theta)d\theta \\
&= -\int \sec(\theta)d\theta + \int -\csc(\theta)\cot(\theta)d\theta.
\end{aligned}$$

Both of the final integrals above we can do. So we get

$$-\int \sec(\theta)d\theta + \int -\csc(\theta)\cot(\theta)d\theta = -\ln\left(\sec(\theta)+\tan(\theta)\right) + \csc(\theta) + D.$$

We now have to draw a triangle. Since $x = \cot(\theta)$, we get that $\tan(\theta) = \frac{1}{x}$ (we could do that without the triangle), and $\sec(\theta) = \frac{\sqrt{x^2+1}}{x}$, $\csc(\theta) = \sqrt{x^2+1}$. So we now have

$$\int \frac{\sqrt{1+x^2}}{x}dx = -\ln\left(\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right) + \sqrt{x^2+1} + D.$$

27.

$$\int \frac{dx}{\left(x^2+2x+2\right)^2}$$

First, we complete the square: gives

$$\int \frac{dx}{\left(x^2+2x+2\right)^2} = \int \frac{dx}{\left(\left(x+1\right)^2+1\right)^2}$$

and then use the substitution $u = x+1$, which gives

$$\int \frac{dx}{\left(\left(x+1\right)^2+1\right)^2} = \int \frac{du}{\left(u^2+1\right)^2}.$$

Now, we use the trigonometric substitution $u = \tan(\theta)$:

$$\begin{aligned}
\int \frac{du}{\left(u^2+1\right)^2} &= \int \frac{\sec^2(\theta)}{\left(\tan^2(\theta)+1\right)^2} d\theta \\
&= \int \frac{\sec^2(\theta)}{\sec^4(\theta)} d\theta \\
&= \int \frac{d\theta}{\sec^2(\theta)} \\
&= \int \cos^2(\theta) d\theta \\
&= \int \frac{1}{2} + \frac{1}{2}\cos(2\theta) d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + D.
\end{aligned}$$

Next, we need to get back to the variable x, so first, we use the identity: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, this gives

$$\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + D = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + D.$$

Now, since $u = \tan(\theta)$, we can draw a triangle to find both $\sin(\theta)$ and $\cos(\theta)$, and we find that $\sin(\theta) = \frac{u}{\sqrt{u^2+1}}$ and $\cos(\theta) = \frac{1}{\sqrt{u^2+1}}$. So

$$\frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + D = \frac{1}{2}\tan^{-1}(u) + \frac{1}{2}\frac{u}{u^2+1} + D.$$

Finally, substituting $u = x+1$ gives

$$\int \frac{du}{\left(u^2+1\right)^2} = \frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\frac{x+1}{(x+1)^2+1} + D.$$

Section 7.4:

25.

$$\int \frac{10}{(x-1)(x^2+9)}dx$$

So partial fractions it is. We guess

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9},$$

and can solve for A by setting $x = 1$. This gives $A = 1$. So now the expression is

$$10 = x^2 + 9 + Bx^2 - Bx + Cx - C.$$

Since this has to hold for all x, this forces $B = -1$ (why?) and hence $C = -1$ as well (again why?). So

$$\int \frac{10}{(x-1)(x^2+9)}dx = \int \frac{1}{x-1} - \frac{x+1}{x^2+9}dx.$$

We now expand:

$$\int \frac{1}{x-1} - \frac{x+1}{x^2+9}dx = \int \frac{1}{x-1}dx - \int \frac{x}{x^2+9}dx - \int \frac{1}{x^2+9}dx.$$

Each one of the above integrals can be done without too much trouble. So we have

$$\int \frac{10}{(x-1)(x^2+9)}dx = \ln(x-1) - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + R.$$

47.

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx$$

Here we set $u = e^x$, and $du = e^x dx$, so we have

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx = \int \frac{e^x}{e^{2x}+3e^x+2}e^x dx = \int \frac{u}{u^2+3u+2}du.$$

The right-most integral above is in the form for partial fractions. So we set:

$$\frac{u}{u^2+3u+2} = \frac{A}{u+2} + \frac{B}{u+1},$$

and find that $B = -1$ and $A = 2$ by setting $u = -1$ and $u = -2$ respectively. So now

$$\int \frac{u}{u^2+3u+2}du = \int \frac{2}{u+2} - \frac{1}{u+1}du = 2\ln(u+2) - \ln(u+1) + D.$$

After resubstitution, we get

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx = 2\ln(e^x+2) - \ln(e^x+1) + D.$$

Part 2: The *fun* problems.

1. Consider the following integral:

$$\int \frac{t^5}{\sqrt{t^2+2}}dt.$$

a) Solve the integral in two ways, first by using the substitution $u = \sqrt{2}\tan(u)$ and then solving the resulting trigonometric integral.

Using the substitution suggested, we have $dt = \sqrt{2}\sec^2(u)du$ and

$$\begin{aligned}
\int \frac{t^5}{\sqrt{t^2+2}}dt &= \int \frac{(\sqrt{2})^5\tan^5(u)}{\sqrt{2}\sqrt{\tan^2(u)+1}}\sqrt{2}\sec^2(u)du \\
&= (\sqrt{2})^5 \int \frac{\tan^5(u)\sec^2(u)}{\sec(u)}du \\
&= (\sqrt{2})^5 \int \tan^5(u)\sec(u)du \\
&= (\sqrt{2})^5 \int \tan^4(u)\sec(u)\tan(u)du \\
&= (\sqrt{2})^5 \int \left(\sec^2(u)-1\right)^2 \sec(u)\tan(u)du
\end{aligned}$$

Now we are ready to set $v = \sec(u)$ and $dv = \sec(u)\tan(u)du$. So we have

$$\begin{aligned}
(\sqrt{2})^5 \int \left(\sec^2(u)-1\right)^2 \sec(u)\tan(u)du &= (\sqrt{2})^5 \int \left(v^2-1\right)^2 dv \\
&= (\sqrt{2})^5 \int v^4 - 2v^2 + 1dv \\
&= (\sqrt{2})^5 \left[\frac{1}{5}v^5 - \frac{2}{3}v^3 + v\right] + D.
\end{aligned}$$

Now we start back substituting. First up was that $v = \sec(u)$. So

$$(\sqrt{2})^5 \left[\frac{1}{5}v^5 - \frac{2}{3}v^3 + v\right] + D = (\sqrt{2})^5 \left[\frac{1}{5}\sec^5(u) - \frac{2}{3}\sec^3(u) + \sec(u)\right] + D.$$

And finally, we have to use the fact that $t = \sqrt{2}\tan(u)$. Or more precisely, $\frac{t}{\sqrt{2}} =$

$\tan(u)$. Drawing a triangle gives that $\sec(u) = \frac{\sqrt{t^2+2}}{\sqrt{2}}$. So our final answer is

$$\int \frac{t^5}{\sqrt{t^2+2}}dt = (\sqrt{2})^5 \left[\frac{1}{5}\frac{(t^2+2)^{\frac{5}{2}}}{(\sqrt{2})^5} - \frac{2}{3}\frac{(t^2+2)^{\frac{3}{2}}}{(\sqrt{2})^3} + \frac{\sqrt{t^2+2}}{\sqrt{2}} \right] + D$$
$$= \frac{1}{5}(t^2+2)^{\frac{5}{2}} - \frac{4}{3}(t^2+2)^{\frac{3}{2}} + 4\sqrt{t^2+2} + D.$$

b) Solve the integral by the method of integration by parts. It might be helpful to rewrite the integral as follows:

$$\int t^4 \frac{t}{\sqrt{t^2+2}}dt,$$

and let $u = t^4$ and $v' = \frac{t}{\sqrt{t^2+2}}$.

So we follow the suggested first step. If $u = t^4$ and $v' = \frac{t}{\sqrt{t^2+2}}$, then $u' = 4t^3$ and $v = \sqrt{t^2+2}$. So we have

$$\int t^4 \frac{t}{\sqrt{t^2+2}}dt = t^4\sqrt{t^2+2} - \int 4t^3\sqrt{t^2+2}dt$$
$$= t^4\sqrt{t^2+2} - 4\int t^2 t\sqrt{t^2+2}dt.$$

Notice that the integral on the second line has been put into the form for another substitution. Here, $u = t^2$ and $v' = t\sqrt{t^2+2}$, making $u' = 2t$ and $v = \frac{1}{3}(t^2+2)^{\frac{3}{2}}$. This gives

$$t^4\sqrt{t^2+2} - 4\int t^3\sqrt{t^2+2}dt = t^4\sqrt{t^2+2} - 4\left[\frac{1}{3}t^2(t^2+2)^{\frac{3}{2}} - \int \frac{2}{3}t(t^2+2)^{\frac{3}{2}}dt\right]$$
$$= t^4\sqrt{t^2+2} - \frac{4}{3}t^2(t^2+2)^{\frac{3}{2}} + \frac{8}{3}\int t(t^2+2)^{\frac{3}{2}}dt.$$

Now the last integral is simple to do, so we now have

$$\int t^4 \frac{t}{\sqrt{t^2+2}}dt = t^4\sqrt{t^2+2} - \frac{4}{3}t^2(t^2+2)^{\frac{3}{2}} + \frac{8}{15}(t^2+2)^{\frac{5}{2}} + D.$$

c) Your two answers, which should both be in terms of the original variable t. If you have done everything correctly, your answer to part a) should look different than your answer to part b). Show that you answer to part a) is indeed equal to your answer to part b).

We now have to compare the following two functions:

$$f_1 = t^4\sqrt{t^2+2} - \frac{4}{3}t^2(t^2+2)^{\frac{3}{2}} + \frac{8}{15}(t^2+2)^{\frac{5}{2}} + D$$

and

$$f_2 = \frac{1}{5}(t^2+2)^{\frac{5}{2}} - \frac{4}{3}(t^2+2)^{\frac{3}{2}} + 4\sqrt{t^2+2} + D.$$

The simplest way to do this is by direct calculation. I.e. we will factor a $\sqrt{t^2+2}$ out of each function. After doing this, the first becomes

$$f_1 = \frac{\sqrt{t^2+2}\left(15\,t^4 - 20\,t^2\left(t^2+2\right) + 8\,\left(t^2+2\right)^2\right)}{15}$$

and the second

$$f_2 = \frac{\sqrt{t^2+2}\left(3\,\left(t^2+2\right)^2 - 20\,t^2 + 20\right)}{15}.$$

Next, we notice that

$$15\,t^4 - 20\,t^2\left(t^2+2\right) + 8\,\left(t^2+2\right)^2 = 3\,\left(t^2+2\right)^2 - 20\,t^2 + 20 = 3t^4 - 8t^2 + 32.$$

So finally, we have that

$$f_1 = f_2 = \frac{\sqrt{t^2+2}}{15}\left(3t^4 - 8t^2 + 32\right).$$

2. Consider the following relationship between x and z:

$$z = \tan\left(\frac{x}{2}\right)$$

Show that under the above relation, one has

$$\begin{cases} \cos(x) = \frac{1-z^2}{1+z^2} \\ \sin(x) = \frac{2z}{1+z^2}. \end{cases}$$

Hint: Using double angle identities will help!

So here, we use the hint:

$$\begin{cases} \cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1 \\ \sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right). \end{cases}$$

Consider the first identity:

$$\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2\left(\frac{x}{2}\right)} - 1$$
$$= \frac{2}{1+\tan^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+z^2} - 1$$
$$= \frac{1-z^2}{1+z^2}.$$

Now the second:

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 1\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \cdot \cos^2\left(\frac{x}{2}\right)$$
$$= 2\tan\left(\frac{x}{2}\right)\frac{1}{\sec^2\left(\frac{x}{2}\right)} = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$
$$= \frac{2z}{1+z^2}.$$

3. Using problem 2, show the following:

$$\int \frac{1}{1+\cos(x)}dx = \tan\left(\frac{x}{2}\right) + C$$

Using problem 2, let $z = \tan\left(\frac{x}{2}\right)$. Then $\cos(x)$ is given as above. However, we must compute dx.

$$x = 2\tan^{-1}(z) \Longrightarrow dx = \frac{2dz}{1+z^2}.$$

Thus, one has

$$\int \frac{1}{1+\cos(x)}dx = \int \frac{1}{1+\frac{1-z^2}{1+z^2}}\frac{2dz}{1+z^2} = \int dz = z + c = \tan\left(\frac{x}{2}\right) + C.$$