Math 2315 - Calculus II

Homework #4 - 2007.09.12

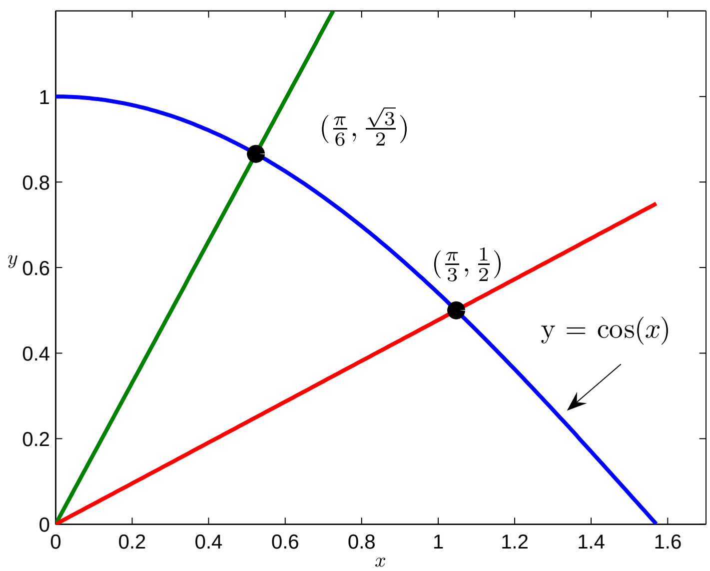
Due Date - 2007.09.19

Solutions

Part 1: Problems from sections 6.1.

Part 2: The *fun* problems.

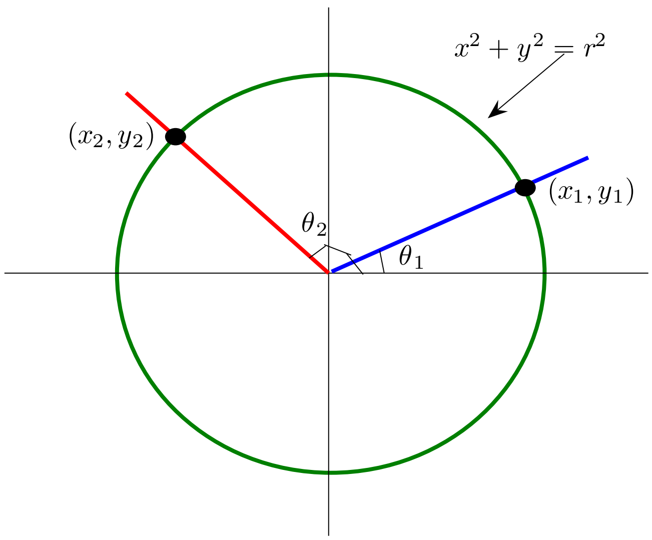
1. Find the area between the curve $y = \cos(x)$ and the two lines depicted below.



The first thing we need to do is determine the equation of the two lines. This is easy, since they go through the origin, they are both of the form $y = mx$, where $m = \frac{y_i}{x_i}$ will give the slopes. So we have $y = \frac{3\sqrt{3}}{\pi}x$ and $y = \frac{3}{2\pi}x$. We now have to set up the integrals. Notice there will be two, since starting at $x = 0$ and moving to the right, we notice the top function is $y = \frac{3\sqrt{3}}{\pi}x$ and the bottom function is $y = \frac{3}{2\pi}x$ until we get to $x = \frac{\pi}{6}$. After $x = \frac{\pi}{6}$, $y = \cos(x)$ is on top and $y = \frac{3}{2\pi}x$ is on bottom. So we have

$$\begin{aligned} Area &= \int_0^{\frac{\pi}{6}} \frac{3\sqrt{3}}{\pi}x - \frac{3}{2\pi}x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(x) - \frac{3}{2\pi}x dx \\ &= \frac{3}{2\pi}\left(\sqrt{3} - \frac{1}{2}\right)x^2\Big|_0^{\frac{\pi}{6}} + \left(\sin(x) - \frac{3}{4\pi}x^2\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\pi}{24}\left(\sqrt{3} - \frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2} - \frac{\pi}{16} - \frac{1}{2}\right) \\ &= \left(\frac{\sqrt{3}}{24} - \frac{1}{12}\right)\pi + \frac{\sqrt{3}}{2} - \frac{1}{2}. \end{aligned}$$

2. Find the area of the circle of radius r between the two lines defined by the angles θ_1 and θ_2 which are measured from the positive $x-$axis. Here you may assume that $0 < \theta_1 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta_2 < \pi$.



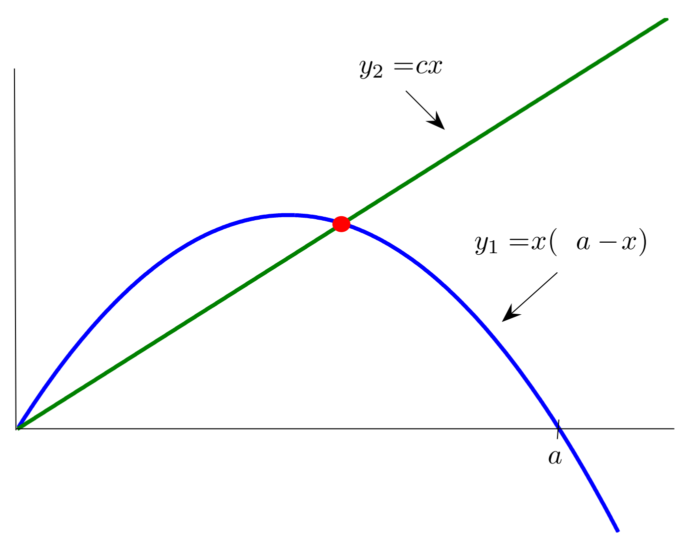
From the point $x = x_2$ to $x = 0$, the circle is on top and the line defined by the angle θ_2 is on bottom. Then from $x = 0$ to $x = x_1$ the line defined by the angle θ_1 is on bottom. So we should first determine the equations of the two lines. They are given by $y_1 = \tan(\theta_1)x$ and $y_2 = \tan(\theta_2)x$. Also, the point x_2 is given in terms of θ_2 by $x_2 = r\cos(\theta_2)$ and similarly $x_1 = r\cos(\theta_1)$. So the area is given by

$$Area = \int_{r\cos(\theta_2)}^{0} \sqrt{r^2 - x^2} - \tan(\theta_2)x dx + \int_0^{r\cos(\theta_1)} \sqrt{r^2 - x^2} - \tan(\theta_1)x dx.$$

Now all we have to do is calculate these integrals! First we rewrite:

$$
\begin{aligned}
Area &= \int_{r\cos(\theta_2)}^{r\cos(\theta_1)} \sqrt{r^2-x^2}dx - \left[\int_{r\cos(\theta_2)}^{0} \tan(\theta_2)\, x dx + \int_0^{r\cos(\theta_1)} \tan(\theta_1)\, x dx\right] \\
&= \left[\frac{x}{2}\sqrt{r^2-x^2} + \frac{1}{2}r^2\sin^{-1}\left(\frac{x}{r}\right)\right]\Bigg|_{r\cos(\theta_2)}^{r\cos(\theta_1)} \\
&\quad - \frac{1}{2}\left[\tan(\theta_2)x^2\Big|_{r\cos(\theta_2)}^{0} + \tan(\theta_1)x^2\Big|_0^{r\cos(\theta_2)}\right] \\
&= \frac{r^2}{2}\left[\cos(\theta_1)\sin(\theta_1) - \cos(\theta_2)\sin(\theta_2) + \sin^{-1}(\cos(\theta_1)) - \sin^{-1}(\cos(\theta_2))\right] \\
&\quad + \frac{r^2}{2}\left[\tan(\theta_2)\cos^2(\theta_2) - \tan(\theta_1)\cos^2(\theta_1)\right] \\
&= \frac{r^2}{2}\left[\sin^{-1}(\cos(\theta_1)) - \sin^{-1}(\cos(\theta_2))\right] \\
&= \frac{r^2}{2}\left[\theta_2 - \theta_1\right]
\end{aligned}
$$

3. Find the value of c such that the area between the curves $y_1 = x(a-x)$ (with $a > 0$) and $y_2 = cx$ is exactly one half of the area between y_1 and the $x-$axis.



The area between y_1 and the $x-$axis is given by

$$B = \int_0^a ax - x^2 dx = \frac{1}{6}a^3.$$

Now we want the area between the two curves to equal $\frac{1}{12}a^3$. So first we set up the integral:

$$A = \int_0^{a-c} x(a-x) - cx dx = \frac{1}{6}(a-c)^3.$$

Setting $\frac{1}{6}(a-c)^3 = \frac{1}{12}a^3$ and solving for c gives $c = a\left(1 - \frac{1}{2^{\frac{1}{3}}}\right)$.