Math 2315 - Calculus II

Homework #5 - 2007.09.17

Due Date - 2007.09.25

Solutions

Part 1: Problems from sections 6.2. and 6.3

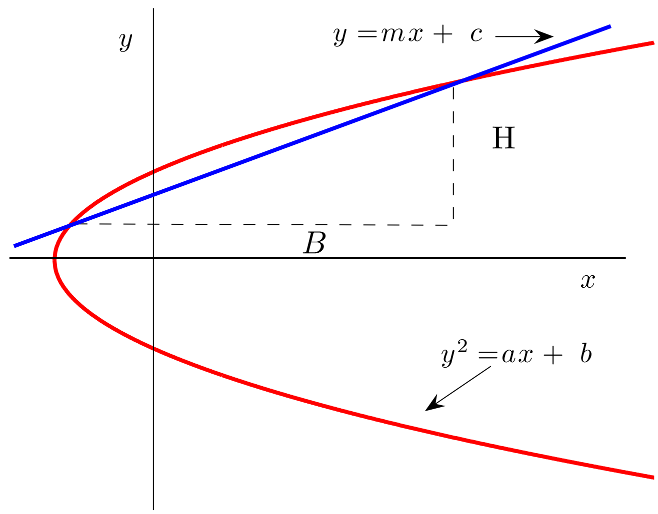
Part 2: The *fun* problems.

1. Verify the formula

$$\int_{x_1}^{x_2} (x - x_1)(x - x_2)dx = \frac{1}{6}(x_1 - x_2)^3$$

$$\begin{aligned}
\int_{x_1}^{x_2} (x - x_1)(x - x_2)dx &= \int_{x_1}^{x_2} x^2 - (x_1 + x_2)x + x_1 x_2 dx \\
&= \frac{1}{3}x^3 - \frac{1}{2}(x_1 + x_2)^2 + x_1 x_2 x\Big|_{x_1}^{x_2} \\
&= \frac{1}{3}x_2^3 - \frac{1}{2}(x_1 + x_2)x_2^2 + x_1 x_2^2 - \frac{1}{3}x_1^3 + \frac{1}{2}(x_1 + x_2)x_1^2 - x_1^2 x_2 \\
&= \frac{1}{6}\left[2x_2^3 - 3x_1 x_2^2 - 3x_2^3 + 6x_1 x_2^2 - 2x_1^3 + 3x_1^3 + 3x_1^2 x_2 - 6x_1^2 x_2\right] \\
&= \frac{1}{6}\left[-x_2^3 + 3x_1 x_2^2 - 3x_1^2 x_2 + x_1^3\right] \\
&= \frac{1}{6}(x_1 - x_2)^3
\end{aligned}$$

2. Using problem 1, show that the area between the two curves given below, revolved around the $x-$ axis, gives a volume $V = \frac{\pi}{6}BH^2$.



Hint: Let x_1 and x_2 be the root of the function $f(x) = ax + b - (mx + c)^2$, where $x_1 < x_2$. Show that

$$V = \pi \int_{x_1}^{x_2} f(x)dx.$$

So first, we write down the integral (where x_1 and x_2 are the roots of intersection):

$$\begin{aligned}
V &= \pi \int_{x_1}^{x_2} \left(\sqrt{ax+b}\right)^2 - (mx+c)^2\,dx \\
&= \pi \int_{x_1}^{x_2} -m^2x^2 + (a - 2mc)x + (b - x^2)dx \\
&= -\pi m^2 \int_{x_1}^{x_2} x^2 - \left(\frac{a-2mc}{m^2}\right)x - \left(\frac{b-c^2}{m^2}\right)dx \\
&= -\pi m^2 \int_{x_1}^{x_2} (x - x_1)(x - x_2)dx = -\frac{\pi m^2}{6}(x_1 - x_2)^3.
\end{aligned}$$

So next we need to determine the values of x_1 and x_2. We need to solve the following quadratic:

$$x^2 - \left(\frac{a-2mc}{m^2}\right)x - \left(\frac{b-c^2}{m^2}\right) = 0,$$

and we get

$$x_1 = \frac{a - 2mc - \sqrt{a^2 - 4amc + 4bm^2}}{2m^2}, x_2 = \frac{a - 2mc + \sqrt{a^2 - 4amc + 4bm^2}}{2m^2}.$$

This also gives

$$y_1 = \frac{a + \sqrt{a^2 - 4amc + 4bm^2}}{2m}, y_2 = \frac{a - \sqrt{a^2 - 4amc + 4bm^2}}{2m}.$$

Next, we have

$$x_1 - x_2 = -\frac{\sqrt{a^2 - 4amc + 4bm^2}}{m^2},$$

so

$$\begin{aligned}
-\frac{\pi m^2}{6}(x_1 - x_2)^3 &= -\frac{\pi m^2}{6}\left[-\frac{\sqrt{a^2 - 4amc + 4bm^2}}{m^2}\right]^3 \\
&= \frac{\pi m^2}{6}\left(\frac{\sqrt{a^2 - 4amc + 4bm^2}}{m^2}\right)^3 \\
&= \frac{\pi}{6m^4}\left(\sqrt{a^2 - 4amc + 4bm^2}\right)^3
\end{aligned}$$

The big question, is does this equal $\frac{\pi}{6}BH^2$? So we need to compute B and H:

$$B = x_2 - x_1 = \frac{\sqrt{a^2 - 4amc + 4bm^2}}{m^2}, H = y_2 - y_1 = \frac{\sqrt{a^2 - 4amc + 4bm^2}}{m}.$$

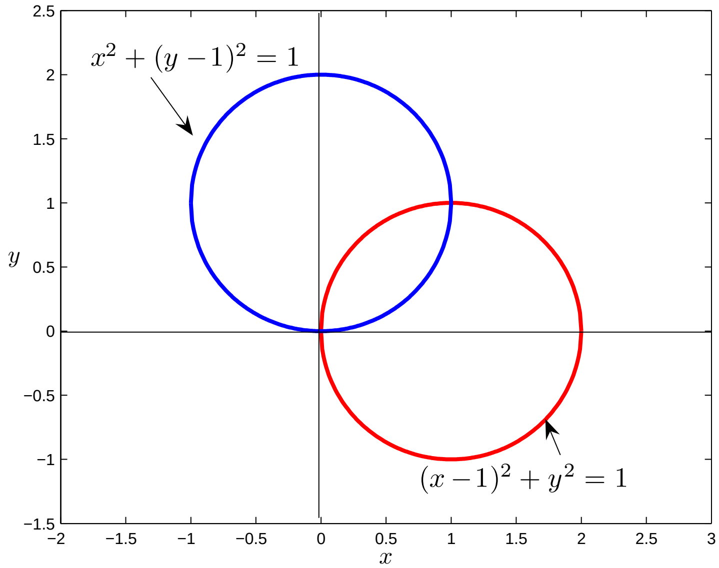
Next, we have that

$$\begin{aligned}
\frac{\pi}{6}BH^2 &= \frac{\pi}{6}\left(\frac{\sqrt{a^2 - 4amc + 4bm^2}}{m^2}\right)\left(\frac{\sqrt{a^2 - 4amc + 4bm^2}}{m}\right)^2 \\
&= \frac{\pi}{6m^4}\left(\sqrt{a^2 - 4amc + 4bm^2}\right)^3
\end{aligned}$$

Indeed, we do find that

$$V = \pi \int_{x_1}^{x_2} \left(\sqrt{ax + b}\right)^2 - (mx + c)^2 \, dx = \frac{\pi}{6}BH^2.$$

3. Let R be the intersection of the circles of radius 1 centered at (1,0) and (0,1). See the picture below.



a) Express the area R as an integral (do not evaluate)

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The top function is given by $y_t = \sqrt{1-(x-1)^2}$, and the bottom function is given by $y_b = 1-\sqrt{1-x^2}$. So we get

$$A = \int_0^1 \sqrt{1-(x-1)^2} - \left(1-\sqrt{1-x^2}\right)\,dx.$$

b) The volume of the revolution of R about the $x-$axis (do not evaluate the integrals).

We will be integrating dx, and the outside radius is given by $r_o = \sqrt{1-(x-1)^2}$ and the inside by $r_I = 1-\sqrt{1-x^2}$. So our integral is

$$V = \int_0^1 \pi\left[\left(\sqrt{1-(x-1)^2}\right)^2 - \left(1-\sqrt{1-x^2}\right)^2\right]dx.$$

c) The volume of the revolution of R about the line $x=-2$ (do not evaluate the integrals).

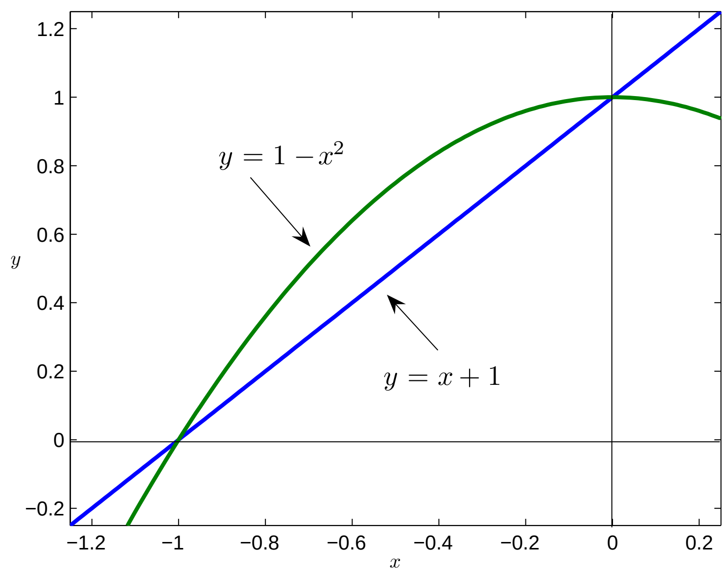
Since we already did all the work for the area between the curves and setting things up for part b), we will use the method of cylindrical shells. We have that

$$V = \int_0^1 2\pi r h dx.$$

First we notice that $r = 2 + x$, and finally, that h was calculated in part a)!. So we have

$$V = \int_0^1 2\pi(2+x)\left(\sqrt{1-(x-1)^2} - \left(1 - \sqrt{1-x^2}\right)\right) dx.$$

4. As a group, devise a method to find the volume of the object generated by taking the area between the curves $y = 1 - x^2$ and $y = x + 1$ and revolving it about the line $y = x + 1$. See the figure below.



Answer will depend upon the approach you take.