# Math 2315 - Calculus II

**Homework #6 - 2007.09.25**
**Due Date - 2007.10.01**
**Solutions**

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Part 1: Problems from sections 6.5. and 7.8

Part 2: The *fun* problems.

1. If for any $x \geq 0$ the average value of $R(x)$ on the interval $[0, x]$ is equal to $x$, find $R(x)$.

We set up the integral:

$$\frac{1}{x-0}\int_0^x R(x)dx = x,$$

and this gives

$$\int_0^x R(x)dx = x^2.$$

Taking a derivative yields $R(x) = 2x$.

2. The average value of $g(t)$ on $[2, 5]$ is equal to 9, find $\int_2^5 g(t)dt$.

First we start with what we know, the average value:

$$\frac{1}{5-2}\int_2^5 g(t)dt = 9,$$

and then we solve for the integral:

$$\int_2^5 g(t)dt = 27.$$

3. Consider the following improper integral:

$$I = \int_4^\infty \frac{dx}{(x-2)(x-3)}.$$

a) Show that if $R > 4$

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left(\left|\frac{R-3}{R-2}\right|\right) - \ln\left(\frac{1}{2}\right).$$

By partial fractions,

$$\begin{aligned}\int_4^R \frac{dx}{(x-2)(x-3)} &= \int_4^R \frac{1}{x-3} - \frac{1}{x-2} dx \\ &= [\ln(|x-3|) - \ln(|x-2|)] \Big|_4^R \\ &= \ln\left(\left|\frac{x-3}{x-2}\right|\right) \Big|_4^R \\ &= \ln\left(\left|\frac{R-3}{R-2}\right|\right) - \ln\left(\frac{1}{2}\right).\end{aligned}$$

b) Show that $I = \ln(2)$.

We take $\lim_{R\to\infty}$ of the above integral:

$$\begin{aligned}I &= \lim_{R\to\infty} \int_4^R \frac{dx}{(x-2)(x-3)} \\ &= \lim_{R\to\infty} \ln\left(\left|\frac{R-3}{R-2}\right|\right) - \ln\left(\frac{1}{2}\right) \\ &= \ln(1) - \ln\left(\frac{1}{2}\right) \\ &= -\ln\left(\frac{1}{2}\right) = \ln(2)\end{aligned}$$

4. Assuming that $a > 0$,

a) Show that $\lim_{x\to\infty} \frac{x^a}{\ln(x)} = \infty$

First, we note that

$$\lim_{x\to\infty} \frac{x^a}{\ln(x)}$$

is in the form $\frac{\infty}{\infty}$. So we can apply L'Hopital's rule:

$$\lim_{x\to\infty} \frac{x^a}{\ln(x)} = \lim_{x\to\infty} \frac{ax^{a-1}}{\frac{1}{x}} = \lim_{x\to\infty} ax^a = \infty.$$

b) Show that $x^a > 2\ln(x)$ for $x$ sufficiently large.

We apply the same idea as part a):

$$\lim_{x\to\infty} \frac{x^a}{2\ln(x)} = \lim_{x\to\infty} \frac{ax^{a-1}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{1}{2}ax^a = \infty.$$

Since the limit was $\infty$, the numerator gets large quicker than the denominator. This gives that $x^a > 2\ln(x)$ for $x$ sufficiently large (say x ¿ N).

c) Show that $e^{-x^a} < x^{-2}$ for $x$ sufficiently large.
Here we assume that $x > N$ from the previous part:

$$x^a > 2\ln(x) \to x^2 > \ln(x^2) \to e^{x^a} > x^2 \to \frac{1}{x^2} > \frac{1}{e^{x^a}} \to e^{-x^a} < x^{-2}.$$

d) Show that $\int_1^\infty e^{-x^a} dx$ converges.

From part c), we have that if $x > N$,

$$0 < e^{-x^a} < x^{-2},$$

thus

$$0 \leq \int_N^\infty e^{-x^a} dx \leq \int_N^\infty x^{-2} dx.$$

Since the integral on the right converges, the integral on the left converges. Thus

$$\leq \int_1^\infty e^{-x^a}$$

converges as well.

5. Let $S$ be the solid obtained by rotating the region below the graph of $y = \frac{1}{x}$ about the $x$-axis for $1 \leq x \leq \infty$.

a) Use the washer method to compute the volume of $S$. Note that the volume is finite even though $S$ is an infinite region.

The formula for volume is quite straight forward in this case:

$$V = \lim_{R\to\infty} \int_1^\infty \pi \frac{1}{x^2} dx.$$

Computing we get

$$V = \pi - \lim_{R\to\infty} \frac{\pi}{R} = \pi.$$

b) It can be shown that the surface area of $S$ is

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}dx.$$

Show that $SA$ is infinite.

We do not even need to compute. Notice that

$$\frac{1}{x} < \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}$$

and since

$$\int_1^\infty \frac{1}{x}dx$$

diverges, we can conclude that the surface area $SA$ of $S$ is infinite.