Math 2315 - Calculus II

Homework #7 - 2007.10.01
Due Date - 2007.10.08
Solutions

Part 1: Problems from sections 8.1 and 8.2

Part 2: The *fun* problems.

1. a) Give a geometric argument which shows that the arc length of the curve $y = x^2$ on the interval $[0, 1]$ is the same as the arc length of the curve $y = \sqrt{x}$ on the same interval.

Since the functions are inverses of each other and intersect at both end points of the interval in question, the arc length of the two curves must be the same.

b) The arc length of the curve $y = x^2$ on the interval $[0, a]$ is equal to the arc length of the curve $y = \sqrt{x}$ on what interval? You might want to use your answer to part a) to help with part b).

The interval should be $\left[0, a^2\right]$.

2. Consider the parabola $y = \alpha (x - \beta)^2 - \gamma$. Construct a method to simplify the arc length integral on the interval $[a, b]$ by moving the parabola to the origin. Be sure to show that your resulting integral is indeed equal to the original.

So first we set up the original integral:

$$L = \int_a^b \sqrt{1 + (2\alpha (x - \beta))^2} dx$$

Notice that there really is no dependence on the vertical shift of γ, so we can automatically move the function back up by γ units. This would be $y_1 = \alpha (x - \beta)^2$. Notice the integral is still the same. Next, we shift the function y_1 to the left β units. This gives $y_2 = \alpha x^2$. The limits of the integral will change to $a - \beta$ to $b - \beta$.

Our new integral is

$$L = \int_{a-\beta}^{b-\beta} \sqrt{1 + (2\alpha z)^2} dz,$$

where z is just a place holder variable to not cause confusion. To show that this is the same as the original arc length integral. We use the substitution $z = x - \beta$. Then $dy = dx$ and when $z = a - \beta$, $x = a$ and with $z = b - \beta$, $x = b$. We have now transformed the integral back to the original.

3. Find the value of a such that the arc length of the *catenary* $y = \cosh(x)$ for $-a \leq x \leq a$ equals 10.

We start with

$$AL = \int_{-a}^{a} \sqrt{1 + \sinh^2(x)} dx = \int_{-a}^{a} \cosh(x) dx = \sinh(x)\Big|_{-a}^{a} = 2\sinh(a).$$

Setting this equal to 10 gives $a = \sinh^{-1}(5)$.

4. Find the arc length of

$$y = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}$$

over the interval $[1, 4]$.

Our integral is

$$AL = \int_{1}^{4} \sqrt{1 + \left(\frac{1}{4}x^3 - \frac{1}{x^3}\right)^2} dx.$$

We do some algebraic manipulation now:

$$\begin{aligned}
\int_1^4 \sqrt{1+\left(\frac{1}{4}x^3-\frac{1}{x^3}\right)^2}dx &= \int_1^4 \sqrt{\frac{1}{2}+\frac{1}{16}x^6+\frac{1}{x^6}}dx \\
&= \int_1^4 \sqrt{\frac{(x^6+4)^2}{(16x^6)}}dx \\
&= \int_1^4 \frac{x^6+4}{4x^3}dx \\
&= \int_1^4 \frac{1}{4}x^3+\frac{1}{x^3}dx \\
&= \left(\frac{1}{16}x^4-\frac{1}{2x^2}\right)\Big|_1^4 \\
&= \frac{525}{32}
\end{aligned}$$

5. Show that the arc length of $y=\ln(f(x))$ for $a \le x \le b$ is given by

$$\int_a^b \frac{\sqrt{(f(x))^2+(f'(x))^2}}{f(x)}dx.$$

We start with:

$$\begin{aligned}
\int_a^b \sqrt{1+\left(\frac{d}{dx}\ln(f(x))\right)^2}dx &= \int_a^b \sqrt{1+\left(\frac{f'(x)}{f(x)}\right)^2}dx \\
&= \int_a^b \sqrt{1+\frac{(f'(x))^2}{(f(x))^2}}dx \\
&= \int_a^b \sqrt{\frac{(f(x))^2+(f'(x))^2}{(f(x))^2}}dx \\
&= \int_a^b \frac{\sqrt{(f(x))^2+(f'(x))^2}}{|f(x)|}dx
\end{aligned}$$

Since the domain of $\ln(f(x))$ is all numbers x such that $f(x)>0$, we have that $|f(x)|=f(x)$. The result is proved.

6. Use problem 5 to compute the arc length of $y=\ln(\sin(x))$ for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

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\begin{aligned}
AL &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sqrt{\sin^2(x) + \cos^2(x)}}{\sin(x)} dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc(x) dx \\
&= \ln\left(|\csc(x) - \cot(x)|\right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \ln(1) - \ln(\sqrt{2} - 1) \\
&= \ln(\sqrt{2} - 1).
\end{aligned}
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