Math 2315 - Calculus II

Homework #8 - 2007.10.10

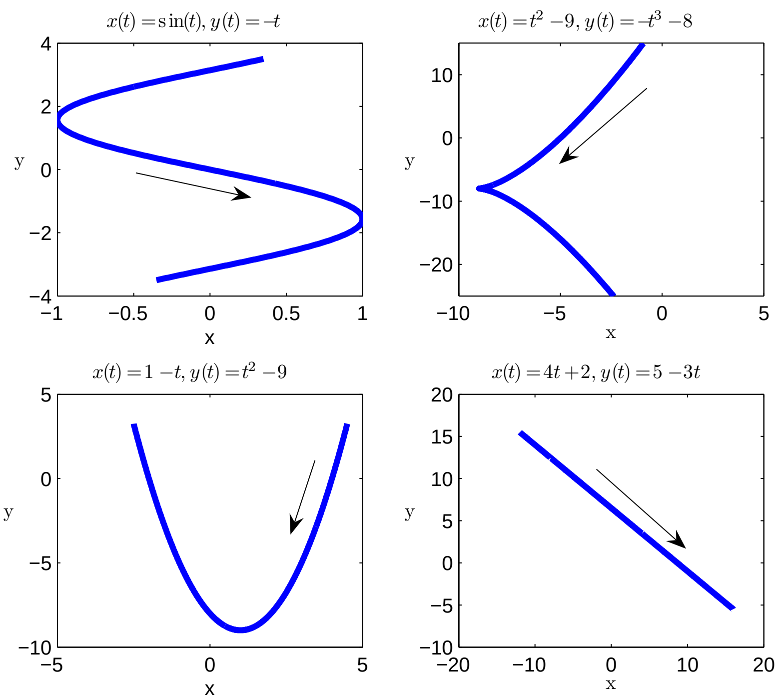
Due Date - 2007.10.17

Solutions

Part 1: Problems from sections 10.1 and 10.2

Part 2: The *fun* problems.

1. Match the parameterizations (a)-(d) below with their plots in the following figure and draw an arrow indicating the direction of motion.



$$a)x(t) = \sin(t), y(t) = -t \qquad b)x(t) = t^2 - 9, y(t) = -t^3 - 8$$
$$c)x(t) = 1 - t, y(t) = t^2 - 9 \qquad d)x(t) = 4t + 2, y(t) = 5 - 3t$$

2. Find a parametrization $c(t) = (x(t), y(t))$ of the line $y = 3x + 4$ such that $c(0) = (2, 10)$.

There are many different ways to do this one (in fact an infinite number of ways), however we will try a simple approach. We will let $y(t) = At+B$ and $x(t) = Ct+D$. Now notice that we want $c(0) = (2, 10)$, so this forces $B = 10$ and $D = 2$. So now we have $y(t) = At + 10$ and $x(t) = Ct + 2$. The final piece of information to use i the fact that $y(t) = 3x(t) + 4$, this gives

$$At + 10 = 3(Ct + 2) + 4 \to A = 3C.$$

So we now have

$$x(t) = Ct + 2, y(t) = 3Ct + 10, \quad c \in \mathbb{R} - \{0\}.$$

We could let $C = 1$ or $C = -2$ etc...

3. Find $\frac{dy}{dx}$ at $t = -4$ if $(x(t), y(t) = \left(t^3, t^2 - 1\right)$.

First, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2 - 1)}{\frac{d}{dt}t^3} = \frac{2t}{3t^2} = \frac{2}{3t},$$

and then substituting in $t = -4$ gives

$$\left.\frac{dy}{dx}\right|_{t=-4} = -\frac{1}{6}.$$

4. If $c(t) = (x(t), y(t)) = \left(\cos(t), \cos(t) + \sin^2(t)\right)$, find an expression for $y = f(x)$ and compute $\frac{dy}{dx}$ both by differentiating $f(x)$ and by differentiating $c(t)$ appropriately.

Since $x = \cos(t)$, we get $\sin^2(t) = 1 - \cos^2(t) = 1 - x^2$. Plugging this into $y = \cos(t) + \sin^2(t)$ gives $y = x + 1 - x^2$.

We can now differentiate $y = f(x)$ easily to get $\frac{dy}{dx} = 1-2x$. We can also compute $\frac{dy}{dx}$ as in problem 3. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\cos(t) + \sin^2(t))}{\frac{d}{dt}\cos(t)} = \frac{-\sin(t) + 2\sin(t)\cos(t)}{-\sin(t)} = 1 - 2\cos(t).$$

Since $x = \cos(t)$, the right hand side of the above string of equalities once again gives $y' = 1 - 2x$.

5. An object moves along the path $x(t) = \frac{1}{4}t^3 + 2t, y(t) = 20t - t^2$ where time is measured in seconds and position in feet.

a) What is the maximum height attained by the object?

To find the maximum height, we maximize $y(t)$, but since it is a parabola which opens down, the maximum height must occur at the vertex, which is $t = 10$ seconds. So $y(10) = 100$ feet.

b) At what time does the object hit the ground?

We have already answered this. We wish to know when $y(t) = 0$. From part a) we know this occurs at $t = 0$ and $t = 20$ seconds. Therefore, at $t = 20$ seconds the object hits the ground.

c) How far is the object from the origin when it hits the ground?

We simply compute $x(20) = 2040$ feet.

6. A bullet fired from a gun follows the trajectory

$$x(t) = at, \quad y(t) = bt - 16t^2 \quad (a, b > 0).$$

Show that the bullet leaves the gun at an angle of $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and lands at a distance $\frac{ab}{16}$ from the origin.

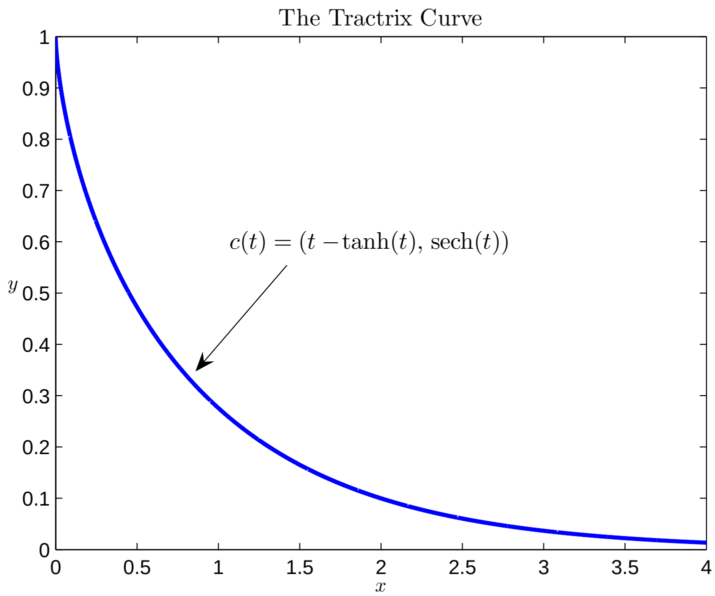
First $x'(t) = a$ and $y'(t) = b - 16t$. Notice that when $t = 0$, $x'(0) = a$ and $y'(0) = b$. Therefore $\left.\frac{dy}{dx}\right|_{t=0} = \frac{b}{a}$. Drawing a right triangle at the origin with adjacent length a and opposite length b gives that the angle θ to be exactly $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ at time $t = 0$.

To determine where the bullet lands, we just set $y(t) = 0$ and find that $t = 0$ (not valid) and $t = \frac{b}{16}$. Plugging this into $x(t)$ gives $x\left(\frac{b}{16}\right) = \frac{ab}{16}$.

7. What is the interpretation of $\sqrt{(x'(t))^2 + (y'(t))^2}$ for a particle following the trajectory $(x(t), y(t))$?

The value $\sqrt{(x'(t))^2 + (y'(t))^2}$ simply gives the *speed* of the particle!

8. The curve $c(t) = (t - \tanh(t), \operatorname{sech}(t))$ is called a *tractrix* (see the figure below). Calculate the surface area of the infinite surface generated by revolving the tractrix about the x-axis for $0 \le t < \infty$.



Surface area is given by

$$SA = \int_0^\infty 2\pi\ y(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt,$$

where

$$x'(t) = 1 - \operatorname{sech}^2(t), \quad y'(t) = -\operatorname{sech}(t)\tanh(t)$$

and thus

$$(x'(t))^2 + (y'(t))^2 = \left(1 - \operatorname{sech}^2(t)\right)^2 + (\operatorname{sech}(t)\tanh(t))^2.$$

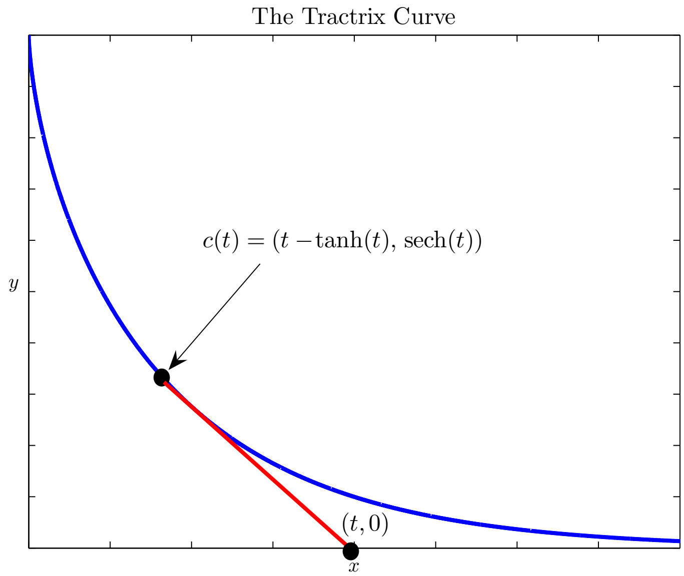
With a little algebraic manipulation (and using the fact that $1 - \operatorname{sech}^2(t) = \tanh^2(t)$ gives

$$(x'(t))^2 + (y'(t))^2 = \tanh^2(t).$$

So now we get

$$SA = \int_0^\infty 2\pi \text{sech}(t)\tanh(t)dt = 2\pi \lim_{R\to\infty} (-\text{sech}(t))\Big|_0^R = 2\pi.$$

9. Verify that the tractrix curve defined in problem 8 has the following property: For all t, the segment from $c(t)$ to $(t,0)$ is tangent to the curve and has length 1. (*See the figure below.*)



First we compute $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\text{sech}(t)\tanh(t)}{1-\text{sech}^2(t)} = -\frac{\text{sech}(t)}{\tanh(t)}.$$

Next, we compute the slope of the line between $c(t)$ and the point $(t,0)$. This is quite simple. At time t, the parametric equation is at the point $(t-\tanh(t), \text{sech}(t))$. So we have two points, and the slope is given by

$$\frac{\Delta y}{\Delta x} = \frac{\text{sech}(t) - 0}{t - \tanh(t) - t} = -\frac{\text{sech}(t)}{\tanh(t)} = \frac{dy}{dx}.$$

We now need to show that the length of the line segment between $c(t)$ and $(t,0)$

has length 1. So we compute

$$\begin{aligned}(\Delta x)^2 + (\Delta y)^2 &= (\operatorname{sech}(t))^2 + (-\tanh(t))^2 \\ &= \operatorname{sech}^2(t) + \tanh^2(t) \\ &= 1 - \tanh^2(t) + \tanh^2(t) = 1.\end{aligned}$$