Math 2315 - Calculus II

Homework #9 - 2007.10.21

Due Date - 2007.10.30

Solutions

Part 1: Problems from sections 10.3 - 10.6

Part 2: The *fun* problems.

1. Let $P_0 = (d, \alpha)$ be polar coordinates of the point on the line L closest to the origin. Show that L has the polar equation $r = d\sec(\theta - \alpha)$.

We start with the equation $y - y_0 = m(x - x_0)$. Here notice that $y_0 = d\sin(\alpha)$ and $x_0 = d\cos(\alpha)$. Since the point (d, α) is closest to the origin on L, the slope of the line connecting the origin to the point (d, α) is given by $\tan(\alpha)$ and that line is perpendicular to the line we are seeking. So in our case, the slope of the line $m = -\cot(\alpha)$. Using the point slope form, we have

$$y - y_0 = -\cot(\alpha)(x - x_0),$$

where $y = r\cos(\theta)$ and $y = r\sin(\theta)$. Plugging everything in we get

$$r\sin(\theta) - d\sin(\alpha) = -\cot(\alpha)(r\cos(\theta) - d\cos(\alpha)).$$

After some simplification we get

$$r\left(\sin(\theta) + \cos(\theta)\cos(\theta)\right) = \frac{d}{\sin(\alpha)}$$

and solving for r after using the difference formula for cosine:

$$r = d\sec(\theta - \alpha).$$

2. Identify the curve with polar equation $r = 2a\cos(\theta)$, where a is constant.

First, we set $r = \sqrt{x^2 + y^2}$ and $\cos(\theta) = \frac{x}{r}$. We get

$$\sqrt{x^2 + y^2} = 2a\frac{x}{\sqrt{x^2 + y^2}},$$

and getting rid of denominators:

$$x^2 + y^2 = 2ax.$$

We complete the square and get

$$(x-a)^2 + y^2 = a^2,$$

which is a circle of a radius a centered at $(a, 0)$.

3. Find the area of the region between the inner and outer loops of the Limaçon defined by $r = 2\cos(\theta) - 1$.

First, notice that since the function is symmetric about the $x-$axis, so we can compute only half the area and multiply the result by 2. So first, to compute half the area of the small loop, we have

$$\int_0^{\frac{\pi}{3}} \frac{1}{2} \left(2\cos(\theta) - 1\right)^2 d\theta,$$

and to compute half the area of the bigger loop,

$$\int_{\frac{\pi}{6}}^{\pi} \frac{1}{2} \left(2\cos(\theta) - 1\right)^2 d\theta.$$

So the total area is is given by

$$\begin{aligned}
A &= 2\left[\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2} \left(2\cos(\theta) - 1\right)^2 d\theta - \int_0^{\frac{\pi}{3}} \frac{1}{2} \left(2\cos(\theta) - 1\right)^2 d\theta\right] \\
&= 2\left[\left(\cos(\theta)\sin(\theta) + \frac{3}{2}\theta - 2\sin(\theta)\right)\Big|_{\frac{\pi}{3}}^{\pi} - \left(\cos(\theta)\sin(\theta) + \frac{3}{2}\theta - 2\sin(\theta)\right)\Big|_0^{\frac{\pi}{3}}\right] \\
&= 3\sqrt{3} + \pi.
\end{aligned}$$

4. If (x, y) has polar coordinates (r, θ), find the coordinates of the following:

a) $(x, -y)$

Polar coordinates are given by $(r, -\theta)$.

b) $(-x, -y)$

Polar coordinates are given by $(r, \theta + \pi)$.

c) $(-x, y)$

Polar coordinates are given by $(r, \pi - \theta)$.

d) (y, x)

Polar coordinates are given by $(r, \frac{\pi}{2} - \theta)$.

e) $(2x, 2y)$

Polar coordinates are given by $(2r, \theta)$.

5. Find the polar equation of the line $y = mx + b$.

Here we set $y = r\sin(\theta)$ and $x = r\cos(\theta)$ and solve for r:

$$r\sin(\theta) = m\ r\cos(\theta) + b \to r = \frac{b}{\sin(\theta) - m\cos(\theta)}.$$

6. Prove that the total area of the four petal rose defined by $r = \sin(2\theta)$ is equal to half the area of the circle of radius 1.

We set up the integral using only one petal and multiplying by four:

$$\begin{aligned} A &= 4\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2(2\theta)d\theta \\ &= 4\left[-\frac{1}{8}\sin(2\theta)\cos(2\theta) + \frac{1}{4}\theta\right]\Big|_0^{\frac{\pi}{2}} \\ &= 4\cdot\frac{\pi}{8} = \frac{\pi}{2}. \end{aligned}$$

7. Why does the circle $r = \sin(\theta)$ lie inside the spiral $r = \theta$? Calculate the area between the two polar curves in the first quadrant.

Since $|\sin(\theta)| \leq |\theta|$ for all θ, the circle must lie inside the spiral.

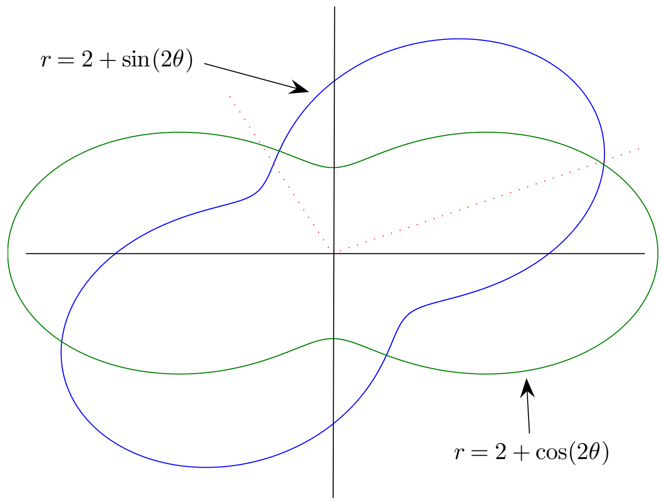
The area is given by

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\theta^2 d\theta - \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2(\theta)d\theta \\ &= \frac{1}{48}\pi^3 - \frac{1}{8}\pi \end{aligned}$$

8. Find the area of the region that lies inside one, but not both of the curves $r = 2 + \sin(2\theta)$ and $r = 2 + \cos(2\theta)$.

First we draw a picture: Notice that all we need to do is calculate the area

Plot of $r = 2 + \sin(2\theta)$ (blue) and $r = 2 + \cos(2\theta)$ (green)



between the two red dashed lines and multiply by 4. So to find the angles of those red lines, we set $2 + \sin(2\theta) = 2 + \cos(2\theta)$. in the first quadrant we have $2\theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{8}$. In the second quadrant we have $2\theta = \frac{5\pi}{4}$ or $\theta = \frac{5\pi}{8}$. So we now set up the integral

$$\begin{aligned} A &= 4\int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2}\left[(2+\sin(2\theta))^2 - (2+\cos(2\theta)2)^2\right] d\theta \\ &= 4\left[-\cos(2\theta) - \frac{1}{4}\cos(2\theta)\sin(2\theta) - \sin(2\theta)\right]\Big|_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \\ &= 8\sqrt{2}. \end{aligned}$$