Math 2315 - Calculus II

Quiz #11 - 2007.11.05

Solutions

Define the sequence $\{a_n\}$ by

$$a_n = \frac{3n^2}{n^2+2}.$$

1. Show that $\{a_n\}$ is strictly increasing.

So we need to show that $a_{n+1} > a_n$ for any n. So we start with the formula for $a_{n+1} > a_n$ and derive a true statement for any $n \in \mathbb{N}$:

$$\begin{aligned}
\frac{3(n+1)^2}{(n+1)^2+2} &> \frac{3n^2}{n^2+2} \\
3(n+1)^2(n^2+2) &> 3n^2\left((n+1)^2+2\right) \\
(n+1)^2n^2 + 2(n+1)^2 &> (n+1)^2n^2 + 2n^2 \\
(n+1)^2 &> n^2.
\end{aligned}$$

2. Find an upper bound for $\{a_n\}$ and prove that it is an upper bound (you do NOT have to use induction).

Since we can naively compute $\lim_{n\to\infty} a_n = 3$, let us try to prove that $\forall n$, $a_n < 3$:

$$\begin{aligned}
\frac{3n^2}{n^2+2} &< 3 \\
3n^2 &< 3(n^2+2) \\
3n^2 &< 3n^2+6 \\
0 &< 6.
\end{aligned}$$

Since the final statement is true, we can work backwards from that statement to get the first statement. Therefore $\forall n$, $a_n < 3$.