Math 2315 - Calculus II

Quiz #13 - 2007.11.28

Solutions

---

1. Find a power series representation for the function $f(x) = \frac{x}{4x+1}$ and determine the interval of convergence.

$$
\begin{aligned}
f(x) &= \frac{x}{4x+1} \\
&= x \cdot \frac{1}{1+4x} \\
&= x \cdot \frac{1}{1-(-4x)} \\
&= x \cdot \sum_{n=1}^{\infty} (-4x)^{n-1} \\
&= x \cdot \sum_{n=1}^{\infty} (-4)^{n-1} x^{n-1} \\
&= \sum_{n=0}^{\infty} -\frac{1}{4}(-4x)^{n-1}
\end{aligned}
$$

This series converges if $|-4x| < 1$ or $-\frac{1}{4} < x < \frac{1}{4}$. So the interval of convergence is $\left(-\frac{1}{4}, \frac{1}{4}\right)$.

2. Find a power series representation for the function $f(x) = \ln(x+3)$ and determine the interval of convergence.

So first, we notice that

$$\frac{d}{dx}\ln(x+3) = \frac{1}{x+3}$$

where

$$\begin{aligned}\frac{1}{x+3} &= \frac{1}{3}\frac{1}{1+\frac{x}{3}} \\ &= \frac{1}{3}\frac{1}{1-(-\frac{x}{3})} \\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n.\end{aligned}$$

So we have

$$\begin{aligned}\ln(x+3) &= \int \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n dx \\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n \frac{1}{n+1}x^{n+1} + D \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{3}\right)^{n+1} + D\end{aligned}$$

Plugging in $x = 0$ gives that $D = \ln(3)$. Therefore

$$\ln(x+3) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{3}\right)^{n+1} + \ln(3).$$

The interval of convergence can be found from the sum

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n,$$

which requires that $\left|-\frac{x}{3}\right| < 1$ or $-3 < x < 3$. So the interval of convergence is $(-3, 3)$.