Math 2315 - Calculus II

Quiz #3 - 2007.09.07

Solutions

Compute the following integral:

$$\int \frac{3x-1}{(x-1)^2(x^2+3)}dx$$

First we need to perform partial fractions.

$$\frac{3x-1}{(x-1)^2(x^2+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+3}.$$

Multiplying both sides by the denominator on the left gives

$$3x-1 = A(x-1)(x^2+3) + B(x^2+3) + (Cx+D)(x^2+3).$$

The one simple value we can plug in is $x = 1$. This gives

$$2 = 4B \longrightarrow B = \frac{1}{2}.$$

Now we use this value for B and move it over to the left hand side to get

$$-\frac{1}{2}x^2 + 3x - 1 = A(x-1)(x^2+3) + (Cx+D)(x^2+3).$$

Now we expand

$$-\frac{1}{2}x^2 + 3x - 1 = A(x^3 - x^2 + 3x - 3) + C(x^3 - 2x^2 + x) + D(x^2 - 2x + 1),$$

and collect like terms:

$$-\frac{1}{2}x^2 + 3x - 1 = (A+C)x^3 + (-A-2C+D)x^2 + (3A+C-2D)x + (-3A+D).$$

We now have the system of equations

$$\begin{aligned} A + C &= 0 \\ -A - 2C + D &= -\frac{1}{2} \\ 3A + C - 2D &= 3 \\ -3A + D &= -\frac{5}{2}, \end{aligned}$$

and using the first equation to get $C = -A$ and plugging this into the next three equations gives

$$\begin{aligned} A + D &= -\frac{1}{2} \\ 2A - 2D &= 3 \\ -3A + D &= -\frac{5}{2}. \end{aligned}$$

Multiplying the first row by 2 and adding it to the second row gives $4A = 2$ or $A = \frac{1}{2}$. This gives $C = -\frac{1}{2}$ and using the equation $-3A + D = -\frac{5}{2}$, we get that $D = -1$. So now we have

$$\begin{aligned} \int \frac{3x-1}{(x-1)^2(x^2+3)}dx &= \int \frac{1}{2}\frac{1}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2} - \frac{1}{2}\frac{x}{x^2+3} - \frac{1}{x^2+3}dx \\ &= \frac{1}{2}\ln\left(|x-1|\right) - \frac{1}{2}\frac{1}{x-1} - \frac{1}{4}\ln\left(x^2+3\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + D. \end{aligned}$$