Math 2315 - Calculus II

Homework #12 Solutions
Assigned - 2010.02.24
Due - 2010.03.01

Textbook problems:

Section 9.2 - 1, 4, 7, 10, 14, 18, 20, 22, 28, 33, 39

Fun Problems:

1. Prove that if a is a positive integer, then

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \frac{1}{a}\left(1 + \frac{1}{2} + \cdots + \frac{1}{a}\right).$$

First, by partial fraction we have that

$$\frac{1}{n(n+a)} = \frac{1}{a}\frac{1}{n} - \frac{1}{a}\frac{1}{n+a},$$

so

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+a)} &= \sum_{n=1}^{\infty} \frac{1}{a}\frac{1}{n} - \frac{1}{a}\frac{1}{n+a} \\ &= \frac{1}{a}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+a} \\ &= \frac{1}{a}\left(1 + \frac{1}{2} + \cdots + \frac{1}{a}\right).\end{aligned}$$

How do we get that last step? Well if you notice, the only terms that will not eventually cancel are those terms of the form $\frac{1}{n}$ for $1 \le n \le a$, since the subtraction term $\frac{1}{n+a}$ starts at $\frac{1}{a+1}$.

2. A ball dropped from a height of 100 ft begins to bounce. Each time it strikes the ground, it returns to two-thirds of its previous height. What is the total distance traveled by the ball if it bounces infinitely many times?

The first time the ball drops, it travels a distance of 100 ft to the ground. Then it travels $100 \cdot \frac{2}{3}$ up and returns the same distance back down and hits the ground again. It then travels a distance of $100 \cdot \left(\frac{2}{3}\right)^2$ back up and back down again etc... So we have that the distance D traveled is given by

$$\begin{aligned} D &= 100 + 2\cdot 100 \cdot \frac{2}{3} + 2\cdot 100\cdot\left(\frac{2}{3}\right)^2 + 2\cdot 100\cdot\left(\frac{2}{3}\right)^3 + \cdots \\ &= 100 + 200\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k \\ &= 100 + 200\left(\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} - 1\right) \\ &= 100 + 200\left(\frac{1}{1-\frac{2}{3}} - 1\right) \\ &= 100 + 200(3-1) \\ &= 500 \text{ ft.} \end{aligned}$$

3. Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n} - \frac{1}{n+a} = \sum_{n=a+1}^{\infty}\frac{1}{n-a} - \frac{1}{n}.$$

We start with the sum

$$\sum_{n=1}^{\infty}\frac{1}{n} - \frac{1}{n+a},$$

and simply replace n with $r-a$ everywhere:

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n} - \frac{1}{n+a} &= \sum_{r-a=1}^{\infty}\frac{1}{r-a} - \frac{1}{r-a+a} \\ &= \sum_{r=a+1}^{\infty}\frac{1}{r-a} - \frac{1}{r}. \end{aligned}$$

Replacing the variable r with n in the final line above gives the result.