Math 2315 - Calculus II

Homework #1 Solutions

Assigned - 2010.01.19

Due - 2010.01.25

Textbook problems:

Section 6.1 - 11-20 all, 21, 26, 27, 29, 30, 31, 34, 35, 36, 37, 43, 44, 53

Fun Problems:

1. Show that if $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\ln(f(x)\,g(x)) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}.$$

$$\begin{aligned}\frac{d}{dx}\ln(f(x)\,g(x)) &= \frac{\frac{d}{dx}(f(x)\,g(x))}{f(x)\,g(x)}\\ &= \frac{f'(x)\,g(x) + f(x)\,g'(x)}{f(x)\,g(x)}\\ &= \frac{f'(x)\,g(x)}{f(x)\,g(x)} + \frac{f(x)\,g'(x)}{f(x)\,g(x)}\\ &= \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}.\end{aligned}$$

2. Since the left hand side of the equation above is symbolically given by:

$$\frac{d}{dx}\ln(f(x)\,g(x)) = \frac{\frac{d}{dx}(f(x)\,g(x))}{f(x)\,g(x)},$$

set the right hand sides of both equations equal to derive a new proof of the product rule $\frac{d}{dx}(f(x)\,g(x))$.

Setting

$$\frac{\frac{d}{dx}(f(x)\,g(x))}{f(x)\,g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

we can solve for $\frac{d}{dx}(f(x)\,g(x))$ by multiplying both sides by the product $f(x)\,g(x)$. This gives

$$\frac{d}{dx}(f(x)\,g(x)) = (f(x)\,g(x))\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right)$$

Upon expanding, we get the definition of the product rule:

$$\frac{d}{dx}(f(x)\,g(x) = f'(x)\,g(x) + f(x)\,g'(x).$$

3. Define $F(x)$ to be the following product:

$$F(x) = \frac{\prod_{i=1}^{n} f_i(x)}{\prod_{j=1}^{m} g_j(x)}.$$

By taking the derivative of $\ln(F(x))$, using the properties of the logarithm, show that

$$F'(x) = F(x) \left[\sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)} - \sum_{j=1}^{m} \frac{g_j'(x)}{g_j(x)} \right].$$

First, notice that the properties of the logarithm yield

$$\ln(F(x)) = \sum_{i=1}^{n} \ln(f_i(x)) - \sum_{j=1}^{m} \ln(g_j(x)).$$

Taking a derivative of both sides yields

$$\frac{F'(x)}{F(x)} = \sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)} - \sum_{j=1}^{m} \frac{g_j'(x)}{g_j(x)},$$

and then solving for $F'(x)$ in the above expression, by multiplying both sides by $F(x)$ gives

$$F'(x) = F(x) \left[\sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)} - \sum_{j=1}^{m} \frac{g_j'(x)}{g_j(x)} \right].$$

4. Use the result of problem 3 to compute the derivative of the following function:

$$F(x) = \frac{(3x-2)(6x+4x^3-2)}{(5x-3)(4x^2-5)(7x^3+3x-1)}$$

Using the formula from problem 3, we get:

$$F'(x) = F(x) \left(\frac{3}{3x-2} + \frac{6+12x^2}{6x+4x^3-2} - \frac{5}{5x-3} - \frac{8x}{4x^2-5} - \frac{21x^2+3}{7x^3+3x-1} \right).$$