Math 3113 - Multivariable Calculus

Homework #2 - 2008.01.23

Due Date - 2008.02.01

Solutions

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1. Prove the Cauchy-Schwarz inequality. If $\overrightarrow{v}$ and $\overrightarrow{w}$ are two vectors, then

$$|\overrightarrow{v} \cdot \overrightarrow{w}| \leq |\overrightarrow{v}|\,|\overrightarrow{w}|.$$

Since we know that

$$\overrightarrow{v} \cdot \overrightarrow{w} = |\overrightarrow{v}|\,|\overrightarrow{w}|\cos(\theta),$$

so

$$\begin{aligned} |\overrightarrow{v} \cdot \overrightarrow{w}| &= ||\overrightarrow{v}|\,|\overrightarrow{w}|\cos(\theta)| \\ &= |\overrightarrow{v}|\,|\overrightarrow{w}|\,|\cos(\theta)| \\ &\leq |\overrightarrow{v}|\,|\overrightarrow{w}| \end{aligned}$$

2. Show that the angle between $\overrightarrow{v} = \langle x, y\rangle$ and $\overrightarrow{v_\theta} = \langle x\cos(\theta) + y\sin(\theta), -x\sin(\theta) + y\cos(\theta)\rangle$ is θ.

We have that

$$\overrightarrow{v} \cdot \overrightarrow{v_\theta} = |\overrightarrow{v}|\,|\overrightarrow{v_\theta}|\cos(\psi),$$

where ψ is the angle between the two vectors. Solving for $\cos(\psi)$ gives

$$\cos(\psi) = \frac{\overrightarrow{v} \cdot \overrightarrow{v_\theta}}{|\overrightarrow{v}|\,|\overrightarrow{v_\theta}|}.$$

We need to compute a few things (work out the details yourself):

$$\overrightarrow{v} \cdot \overrightarrow{v_\theta} = \left(x^2 + y^2\right)\cos(\theta)$$

$$|\overrightarrow{v}| = \sqrt{x^2 + y^2}, \quad |\overrightarrow{v_\theta}| = \sqrt{x^2 + y^2}.$$

Putting this all together gives

$$\cos(\psi) = \frac{\left(x^2 + y^2\right)\cos(\theta)}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2}} = \cos(\theta).$$

So the angle is θ.

3. Prove that $\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{v} \times \overrightarrow{u}$ if and only if $\overrightarrow{u} = \overrightarrow{w} + \lambda\overrightarrow{v}$ for some scalar λ. You may assume that $\overrightarrow{v} \neq \overrightarrow{0}$.

We start proving right to left:

$$\overrightarrow{u} = \overrightarrow{w} + \lambda\overrightarrow{v} \Longrightarrow \overrightarrow{v} \times \overrightarrow{u} = \overrightarrow{v} \times (\overrightarrow{w} + \lambda\overrightarrow{v})$$

Now we work on the new equation:

$$\begin{aligned} \overrightarrow{v} \times \overrightarrow{u} &= \overrightarrow{v} \times (\overrightarrow{w} + \lambda\overrightarrow{v}) \\ &= \overrightarrow{v} \times \overrightarrow{w} + \overrightarrow{v} \times (\lambda\overrightarrow{v}) \\ &= \overrightarrow{v} \times \overrightarrow{w} + \lambda\overrightarrow{v} \times \overrightarrow{v} \\ &= \overrightarrow{v} \times \overrightarrow{w} + \overrightarrow{0} \\ &= \overrightarrow{v} \times \overrightarrow{w} \end{aligned}$$

To go the other way, we simply follow the lines back up!

4. Find the nearest point to $Q = (-1, 3, -1)$ on the plane $x - 4z = 2$.

The first thing we need is the normal vector to the plane $x - 4z = 2$. This is given by $\overrightarrow{n} = \langle 1, 0, -4\rangle$. We can now compute the equation of the line through the point Q which is perpendicular to the plane $x - 4z = 2$. The equations is given in vector form by $Q + t\overrightarrow{n} = (-1 + t, 3, -1 - 4t)$. To find the time at which this line hits the plane, we simply use the fact that $x = -1 + t, y = 3$ and $z = -1 - 4t$ and plug it into the equation of the plane. This gives $-1 + t - 4(-1 - 4t) = 2$. Solving for t gives $t = -\frac{1}{17}$. Plugging this back in to the equation of the line gives the point of intersection to be $\left(-1 - \frac{1}{17}, 3, -1 + \frac{4}{17}\right) = \left(-\frac{18}{17}, 3, -\frac{13}{17}\right)$.

5. The plane $3x - 4y + 2z = 8$ intersects the xy-plane in a line. What is the equation of this line?

The xy-plane is given by $z = 0$, so setting $z = 0$ on the plane gives $3x - 4y = 8$. We can find one point on the plane, simply given by $(0, -2, 0)$. Notice that we have

$$3x - 4y = 8 \longrightarrow 3x - 4(y + 2) = 0 \longrightarrow 3x = 4(y + 2) \longrightarrow \frac{x}{4} = \frac{y + 2}{3}.$$

So the symmetric equation of our line is

$$\frac{x}{4} = \frac{y + 2}{3}, z = 0.$$

6. If $\overrightarrow{e_\theta} = \langle \cos(\theta), \sin(\theta)\rangle$, show $\overrightarrow{e_\theta} \cdot \overrightarrow{e_\psi} = \cos(\theta - \psi)$.

$$\overrightarrow{e_\theta} \cdot \overrightarrow{e_\psi} = \cos(\theta)\cos(\psi) + \sin(\theta)\sin(\psi) = \cos(\theta - \psi).$$

7. Find two vectors which are orthogonal to the vector $\langle -2, 1, 3\rangle$.

Answers will vary here. The idea though, is to find one first, then compute the cross product to get the third. So first we find one vector. To make life easy, I will choose the vector $\langle 1, 2, 0\rangle$. The third vector is $\langle -2, 1, 3\rangle \times \langle 1, 2, 0\rangle = \langle -6, 3, -5\rangle$.