Math 3113 - Multivariable Calculus

Homework #7 - 2008.03.26

Due Date - 2008.04.02

Solutions

1. Calculate the gradients of the following functions.

a) $f(r,s) = \sin(r+s)e^{rs}$

$$\nabla f(r,s) = \langle \cos(r+s)e^{rs} + s\ \sin(r+s)e^{rs}, \cos(r+s)e^{rs} + r\ \sin(r+s)e^{rs}\rangle$$

b) $g(s,t) = \frac{st}{s^2+t^3}$

$$\nabla g(s,t) = \langle \frac{t}{s^2+t^3} - 2\frac{s^2t}{(s^2+t^3)^2}, \frac{s}{s^2+t^3} - 3\frac{st^3}{(s^2+t^3)^2}\rangle$$

c) $h(x,y,z) = 5xye^{-2y^2-3z^2}$

$$\nabla h(x,y,z) = \langle 5ye^{-2y^2-3z^2}, 5xe^{-2y^2-3z^2} - 20xy^2e^{-2y^2-3z^2}, -30xyze^{-2y^2-3z^2}\rangle$$

2. Find the directional derivative of the following functions in the direction $\overrightarrow{v}$ at the point P.

a) $f(x,y) = 2x^2y + xy^3$ at $P = (0,1)$ in the direction $\overrightarrow{v} = \langle 1,-1\rangle$.

First we compute the gradient:

$$\nabla f(x,y) = \langle 4xy + y^3, 2x^2 + 3xy^2\rangle \longrightarrow \nabla f(0,1) = \langle 1,0\rangle.$$

Next, we find a unit vector $\overrightarrow{u}$ in the direction of $\overrightarrow{v}$. First we find the length of $\overrightarrow{v}$, which is $|\overrightarrow{v}| = \sqrt{2}$ and then set $\overrightarrow{u} = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$. Finally, we have

$$D_{\overrightarrow{u}} f(0,1) = \frac{1}{\sqrt{2}}\ \langle 1,0\rangle \cdot \langle 1,-1\rangle = \frac{1}{\sqrt{2}}.$$

b) $g(x,y) = \sin(x-y)$ at $P = \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ in the direction $\overrightarrow{v} = \langle 2,1\rangle$.

First we compute the gradient:

$$\nabla g(x,y) = \langle \cos(x-y), -\cos(x-y)\rangle \longrightarrow \nabla g\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle.$$

Next, we find a unit vector $\overrightarrow{u}$ in the direction of $\overrightarrow{v}$. First we find the length of $\overrightarrow{v}$, which is $|\overrightarrow{v}| = \sqrt{5}$ and then set $\overrightarrow{u} = \frac{1}{\sqrt{5}}\langle 2,1\rangle$. Finally, we have

$$D_{\overrightarrow{u}} g\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{5}}\ \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle \cdot \langle 2,1\rangle = \frac{1}{\sqrt{10}}.$$

c) $h(x,y,z) = ze^{xy} + xz + y^2$ in the direction $\overrightarrow{v} = \langle 2,-1,1\rangle$ at the point $P = (1,0,-1)$.

First we compute the gradient:

$$\nabla h(x,y,z) = \langle yze^{xy} + z, xze^{xy} + 2y, e^{xy} + x\rangle \longrightarrow \nabla h(1,0,-1) = \langle -1,-1,2\rangle.$$

Next, we find a unit vector $\overrightarrow{u}$ in the direction of $\overrightarrow{v}$. First we find the length of $\overrightarrow{v}$, which is $|\overrightarrow{v}| = \sqrt{6}$ and then set $\overrightarrow{u} = \frac{1}{\sqrt{6}}\langle 2,-1,1\rangle$. Finally, we have

$$D_{\overrightarrow{u}} h(1,0,-1) = \frac{1}{\sqrt{6}}\ \langle -1,-1,2\rangle \cdot \langle 2,-1,1\rangle = \frac{1}{\sqrt{6}}.$$

3. Consider the function $f(x,y) = e^{-(x^2+y^2)}$. Prove that the path of steepest ascent for the function at ANY point always points towards the origin.

There are at least two ways to do this, we will do it the computational way. First we compute $\nabla f(x,y)$:

$$\nabla f(x,y) = \langle -2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)} \rangle.$$

Next we perform some simplifications and factoring to the gradient:

$$\nabla f(x,y) = \langle -2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)} \rangle = (-2)e^{-(x^2+y^2)} \langle x, y \rangle$$

Finally, we note that given the point (a,b) the slope of the line through the origin to (a,b) is given by (in vector form) $\langle a, b \rangle$. Notice that it can also be given by $c\langle a, b \rangle$ where c is an arbitrary nonzero constant. Since in our case, we have that $c = (-2)e^{-x^2+y^2}$, we know that the vector lies on the line that passes through the point (a,b) and the origin. To show that it point in the direction towards the origin (and not away), we simply note that $c = (-2)e^{-(x^2+y^2)} < 0$.

4. Find the equation of the tangent plane to the surface $4x^2y^2+6xyz-12yz^3 = 28$ at the point $(x,y,z) = (1,2,-1)$.

First, we set $G(x,y,z) = 4x^2y^2 + 6xyz - 12yz^3$ and compute

$$\nabla G(x,y,z) = \langle 8xy^2 + 6yz, 8x^2y + 6xz - 12z^3, 6xy - 36yz^2 \rangle \longrightarrow \nabla G(1,2,-1) = \langle 20, 22, -60 \rangle.$$

So the equation of the plane is given by

$$20(x-1) + 22(y-2) + 60(z+1) = 0.$$

5. Show that every normal line to the sphere $x^2+y^2+z^2 = r^2$ passes through the center of the sphere.

First, we note that the center is located at the origin. The gradient of the sphere function $F(x,y,z) = x^2+y^2+z^2$ for any level surface r^2 is simply

$$\nabla F(x,y,z) = \langle 2x, 2y, 2z \rangle.$$

The symmetric equations of the normal line at the point (x_0, y_0, z_0) are given by

$$\frac{x-x_0}{2x_0} = \frac{y-y_0}{2y_0} = \frac{z-z_0}{2z_0},$$

or the parametric version is

$$x = x0 + 2x_0t, \quad y = y0 + 2y_0t, z = z0 + 2z_0t.$$

In the symmetric equation case, notice that plugging in $(x,y,z) = (0,0,0)$ satisfies the equations. In the parametric case notice that at time $t = -\frac{1}{2}$, $x = y = z = 0$.