Math 4982 - Senior Seminal

Problem #37

Name: ______________________________

Find

$$\sum_{k=1}^{\infty} \frac{k^2}{k!}.$$

First, we remember that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and therefore,

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Now back to our problem:

$$\sum_{k=1}^{\infty} \frac{k^2}{k!} = \sum_{k=1}^{\infty} \frac{k}{(k-1)!},$$

and by setting $k = k+1$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k}{(k-1)!} &= \sum_{k=0}^{\infty} \frac{k+1}{k!} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} + \frac{k}{k!} \right) \end{aligned}$$

We can break up the last infinite series into two infinite series since they are both convergent on their own:

$$\sum_{k=0}^{\infty} \left(\frac{1}{k!} + \frac{k}{k!} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} + \sum_{k=0}^{\infty} \frac{k}{k!}.$$

Notice that the first term on the right sum is zero, so

$$\sum_{k=0}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}.$$

Therefore, replacing k by $k+1$ in the last sum above, and replacing that back into our original string of equalities gives

$$\sum_{k=1}^{\infty} \frac{k^2}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} + \sum_{k=0}^{\infty} \frac{1}{k!} = 2e.$$