Math 4133 - Linear Algebra

Midterm - Written Portion

Solutions

Consider the system of equations for problems 1–6.

$$x_1 - 2x_2 + 3x_3 + 4x_4 = 1$$
$$-3x_1 + 4x_2 + 6x_3 - 2x_4 + \frac{5}{2}x_5 = 5$$
$$-5x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 = 11$$

1. Without solving, what is the most likely value for the dimension of the solution to the above system.

   Generally, we have that the dimension should be $5 - 3 = 2$.

2. Convert the system into an augmented matrix.

   In augmented form, we have

$$\begin{bmatrix} 1 & -2 & 3 & 4 & 0 & 1 \\ -3 & 4 & 6 & -2 & 5/2 & 5 \\ -5 & 6 & 2 & 3 & 5 & 11 \end{bmatrix}$$

3. Row reduce the matrix from problem 2.

   In row reduced form we have

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{114}{13} & -\frac{5}{2} & -7 \\ 0 & 1 & 0 & -\frac{175}{26} & -\frac{5}{4} & -4 \\ 0 & 0 & 1 & -\frac{3}{13} & 0 & 0 \end{bmatrix}$$

4. Write your solution $(x_1, x_2, x_3, x_4, x_5)$ in terms of the variables x_4 and x_5.

   Using the row reduced matrix from the previous problem, we have that

$$(x_1, x_2, x_3, x_4, x_5) = \left(-7 + \frac{114}{13}x_4 + \frac{5}{2}x_5,\ -4 + \frac{175}{26}x_4 + \frac{5}{4}x_5,\ \frac{3}{13}x_4,\ x_4,\ x_5\right)$$

5. Write your solution $(x_1, x_2, x_3, x_4, x_5)$ in terms of the variables x_3 and x_5.

   There are numerous ways to do this, but consider what happens if we swap the x_3 and x_4 columns in our initial augmented matrix. We would have

$$\begin{bmatrix} 1 & -2 & 4 & 3 & 0 & 1 \\ -3 & 4 & -2 & 6 & 5/2 & 5 \\ -5 & 6 & 3 & 2 & 5 & 11 \end{bmatrix}$$

Here the third column corresponds to x_4 and the fourth column corresponds to x_3. If we row reduce now, we get

$$\begin{bmatrix} 1 & 0 & 0 & -38 & -\frac{5}{2} & -7 \\ 0 & 1 & 0 & -\frac{175}{6} & -\frac{5}{4} & -4 \\ 0 & 0 & 1 & -\frac{13}{3} & 0 & 0 \end{bmatrix}$$

   Our solution now can be written as

$$(x_1, x_2, x_3, x_4, x_5) = \left(-7 + 38x_3 + \frac{5}{2}x_5,\ -4 + \frac{175}{6}x_3 + \frac{5}{4}x_5,\ x_3,\ \frac{13}{3}x_3,\ x_5\right)$$

6. Is it possible to express the solution to this system in terms of any two variables? If not, give at least one example of a pair of variables for which the solution can not be written in terms of.

Since $x_3 = 13/3x_4$, we must have at least one of the variables x_3 or x_4 as independent in any solution. Thus, we could not write our solution in terms of x_1 and x_2 only, x_1 and x_5 only, or x_2 and x_5 only.

Consider the following matrices for problems 7–14.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

7. Compute the determinant of A by expansion along the second column.

$$\begin{aligned}
\det\left(\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}\right) \\
&= (-1)^{1+2} \cdot 2 \cdot \det\left(\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}\right) + (-1)^{2+2} \cdot 1 \cdot \det\left(\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\right) + (-1)^{3+2} \cdot 5 \cdot \det\left(\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\right) \\
&= -1 \cdot 2 \cdot (0 - 0) + 1 \cdot 1 \cdot (3 - 2) - 1 \cdot 5 \cdot (0 - 0) \\
&= 1
\end{aligned}$$

8. Compute the determinant of A by expansion along the second row.

$$\begin{aligned}
\det\left(\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}\right) \\
&= (-1)^{2+1} \cdot 0 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 5 & 1 \end{bmatrix}\right) + (-1)^{2+2} \cdot 1 \cdot \det\left(\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\right) + (-1)^{2+3} \cdot 0 \cdot \det\left(\begin{bmatrix} 3 & 2 \\ -2 & 5 \end{bmatrix}\right) \\
&= -1 \cdot 0 \cdot (2 + 5) + 1 \cdot 1 \cdot (3 - 2) - 1 \cdot 0 \cdot (15 + 4) \\
&= 1
\end{aligned}$$

9. Compute the cofactor matrix C for A.

$$C = \begin{bmatrix} 1 & 0 & 2 \\ -7 & 1 & -19 \\ 1 & 0 & 3 \end{bmatrix}$$

10. Compute the transpose to the cofactor matrix C from problem 9.

$$C^T = \begin{bmatrix} 1 & -7 & 1 \\ 0 & 1 & 0 \\ 2 & -19 & 3 \end{bmatrix}$$

11. Compute the inverse to A using the previous problems.

Since the determinant of A is one, we have that $A^{-1} = C^T$, and C^T is given in problem 10.

12. Compute the LU factorization of A.

A straight-forward approach would be to simply zero out entries $A_{3,1}$ and $A_{3,2}$. So we get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2/3 & 19/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

13. Solve $AX = B$ using the LU factorization from problem 12.

Since $A = LU$, we solve $LUX = B$. To do this, we first let $Y = UX$ and solve $LY = B$ and then $UX = Y$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2/3 & 19/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The first equation is $y_1 = -1$, the second is $y_2 = 1$. The third is $-2/3y_1 + 19/3y_2 + y_3 = 1$. But since we know y_1 and y_2 already, we have $y_3 = -6$. Now we solve $UX = Y$:

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix}$$

The last equation yields $1/3x_3 = -6$, or $x_3 = -18$. The second to last gives $x_2 = 1$. The top equation is $3x_1 + 2x_2 - x_3 = -1$, and since we know x_3 and x_2, we get $x_1 = -7$.

14. Solve $AX = B$ by computing $A^{-1}B$. Verify that this approach yields the same answer as that of problem 13.

$$\begin{bmatrix} 1 & -7 & 1 \\ 0 & 1 & 0 \\ 2 & -19 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \\ -18 \end{bmatrix}$$

Consider the vector $\vec{t} = \langle 1, -2, 1 \rangle$ for problems 15–18.

15. Find a unit vector $\vec{u}$ in the direction of $\vec{t}$.

First, we compute $\left|\vec{t}\right| = \sqrt{1^2 + 2^2 + 1} = \sqrt{6}$. Then, we define

$$\vec{u} = \frac{1}{\sqrt{6}}\vec{t} = \left\langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

16. Find two vectors, $\vec{v}$ and $\vec{w}$, both of which are unit length and satisfy $\vec{u} \perp \vec{v}$, $\vec{u} \perp \vec{w}$ and $\vec{v} \perp \vec{w}$.

There is no one correct answer here. Any two vectors not in the same direction, but lying in the same plane perpendicular to $\vec{u}$, will suffice. So we find one first, and verify with the dot product that it is indeed true. An easy guess is a unit vector in the direction $\langle 1, 0, -1 \rangle$:

$$\vec{v} = \left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$$

Finally, the last vector is simply $\vec{w} = \vec{u} \times \vec{v}$, which is given by

$$\vec{w} = \det\left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}\right) = \left\langle \frac{2}{\sqrt{12}}, \frac{2}{\sqrt{12}}, \frac{2}{\sqrt{12}} \right\rangle$$

Since the vectors $\vec{u}$ and $\vec{v}$ are of unit length, $\vec{w}$ is as well.

17. Define $B \in \mathbb{R}^{3\times 3}$ to be the matrix whose rows are the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ from problems 15 and 16. Perform the matrix multiplication $B\,B^T$.

So we have

$$B = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}} \end{bmatrix}, \quad B^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{12}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{2}{\sqrt{12}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{12}} \end{bmatrix}$$

And finally,

$$B\,B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

18. Come up for a good reason as to why you should have known your answer to problem 17.

Each entry in the matrix $B\,B^T$ corresponds to multiplication of a row of B by a column of B^T. However, each row of B is a vector from the previous problems, and the same goes for the columns of B^T. Thus, we are performing dot products. If we are on the diagonal, we are computing the norm of a vector (squared). Off-diagonal, we have the dot product of perpendicular vectors, which must be zero.

19. Find the distance between the two parallel planes:

$$-5x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 = 15$$
$$-5x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 = -3$$

We will find a point on the first plane. By setting $x_1 = x_2 = x_3 = x_4 = 0$, we get $x_5 = 3$. So now we have a point $(0, 0, 0, 0, 3)$, and a plane $-5x_1 + 6x_2 + 2x_3 + 3x_4 + 5x_5 = -3$. We use equation 6.22, to get

$$D = \frac{|5 \cdot 3 + 3|}{\sqrt{5^2 + 6^2 + 2^2 + 3^2 + 5^2}} = \frac{18}{\sqrt{99}}$$

Consider the vectors $\overrightarrow{u} = \langle 1, 1, 1\rangle$ and $\overrightarrow{v} = \langle -1, 1, 0\rangle$ for problems 20–21.

20. Find the equation of the plane in which the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ lie.

To find the plane $ax + by + cz = d$, we find the normal vector to the plane, which is the cross product of $\overrightarrow{u}$ and $\overrightarrow{v}$. This gives

$$\overrightarrow{n} = \langle 1, 1, 1\rangle \times \langle -1, 1, 0\rangle = \langle -1, -1, 2\rangle$$

Since the plane goes through the origin, we have that $d = 0$. The equation becomes $-x - y + 2z = 0$.

21. Project the vector $\overrightarrow{w} = \langle 4, 5, -4\rangle$ onto the plane from problem 20.

We will do this two ways, and hopefully they will agree. The first is to project $\overrightarrow{w}$ onto $\overrightarrow{n}$, which will give us the portion of $\overrightarrow{w}$ in the normal direction to the plane. The vector we want will be $\text{proj}_{\overrightarrow{u},\overrightarrow{v}}(\overrightarrow{w}) = \overrightarrow{w} - \text{proj}_{\overrightarrow{n}}(\overrightarrow{w})$. So first, we compute $\text{proj}_{\overrightarrow{n}}(\overrightarrow{w})$:

$$\text{proj}_{\overrightarrow{n}}(\overrightarrow{w}) = \frac{\overrightarrow{n} \cdot \overrightarrow{w}}{|\overrightarrow{n}|^2}\overrightarrow{n} = \left\langle \frac{17}{6}, \frac{17}{6}, -\frac{17}{3} \right\rangle$$

Then

$$\text{proj}_{\overrightarrow{u},\overrightarrow{v}}(\overrightarrow{w}) = \langle 4, 5, -4\rangle - \left\langle \frac{17}{6}, \frac{17}{6}, -\frac{17}{3} \right\rangle = \left\langle \frac{7}{6}, \frac{13}{6}, \frac{5}{3} \right\rangle$$

If we project the vector $\overrightarrow{w}$ onto $\overrightarrow{u}$, and then separately onto $\overrightarrow{v}$, the resulting sum of the two projections will give us the desired result. This can be done since $\overrightarrow{u}$ and $\overrightarrow{v}$ are perpendicular.

$$\text{proj}_{\overrightarrow{u}}(\overrightarrow{w}) = \frac{\overrightarrow{u} \cdot \overrightarrow{w}}{|\overrightarrow{u}|^2}\overrightarrow{u} = \left\langle \frac{5}{3}, \frac{5}{3}, \frac{5}{3} \right\rangle, \quad \text{proj}_{\overrightarrow{v}}(\overrightarrow{w}) = \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{|\overrightarrow{v}|^2}\overrightarrow{v} = \left\langle -\frac{1}{2}, \frac{1}{2}, 0 \right\rangle$$

So now we add the two above vectors to get

$$\text{proj}_{\overrightarrow{u},\overrightarrow{v}}(\overrightarrow{w}) = \left\langle \frac{7}{6}, \frac{13}{6}, \frac{5}{3} \right\rangle$$