Math 4133 - Linear Algebra

Final Exam

Name: ________________________________

Consider the two vectors $\overrightarrow{v_1} = \langle 1,0,1,0\rangle$ and $\overrightarrow{v_2} = \langle 1,0,-1,0\rangle$ for problems 1–4.

1. If $\mathbb{S} = \text{span}\,\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$, compute $\mathbb{S}^\perp$ and express your answer as the span of vectors.

2. Project the vector $\overrightarrow{v} = \langle 1,2,3,4\rangle$ onto $\mathbb{S}$.

3. Project the vector $\overrightarrow{v} = \langle 1,2,3,4\rangle$ onto $\mathbb{S}^\perp$.

4. Add the projections from problems 2 and 3 together, what should you expect to get?

Consider the three vectors $\overrightarrow{u_1} = \langle 1,0,1\rangle$, $\overrightarrow{u_2} = \langle 1,-1,0\rangle$, and $\overrightarrow{u_3} = \langle 1,1,1\rangle$ for problems 5–7.

5. Does span span $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\} = \mathbb{R}^3$? Justify your answer.

6. Express the vector $\overrightarrow{u} = \langle 1,2,3\rangle$ in terms of the basis $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$

7. Perform the Gram-Schmidt orthonormalization process on the vectors $\overrightarrow{u_1}$, $\overrightarrow{u_2}$ and $\overrightarrow{u_3}$.

Consider the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ defined by $T(\langle x,y,z\rangle) = \langle x+y+z, 0, x-y+3z, 0\rangle$ for problems 8–12.

8. Compute the matrix A such that $T(\overrightarrow{v}) = A\overrightarrow{v}$.

9. Find a basis for $\text{Im}(T)$. What is the dimension of $\text{Im}(T)$?

10. Find a basis for $\text{Ker}(T)$. What is the dimension of $\text{Ker}(T)$?

11. Let $\mathbf{B}_3 = \{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ be as defined for problems 5–7, and $\mathbf{B}_4 = \{\langle 1,0,1,0\rangle, \langle 1,0,-1,0\rangle, \langle 0,1,0,1\rangle, \langle 0,-1,0,1\rangle\}$ be a basis for $\mathbb{R}^4$. Draw a commutative diagram for the map T' which corresponds to the map T using the bases $\mathbf{B}_3$ and $\mathbf{B}_4$.

12. Using the diagram from problem 11, express the map T' as a matrix A' which is the product of three matrices corresponding to the diagram. You do NOT have to compute the inverse to any matrix.

13. Consider a set of data points $\{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$ and the equation of an ellipse $x^2/a^2 + y^2/b^2 = 1$. If we set $\alpha = 1/a^2$ and $\beta = 1/b^2$, then the equation can be written as $\alpha x^2 + \beta y^2 = 1$. Use this definition to set up a system of equations that will find the values of α and β to best fit the data.

14. For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 5 & 4 & -2 \end{bmatrix}$, find the matrices Q and D such that D is diagonal, Q is invertible, and that $A = QDQ^{-1}$.