CS 3513 - Numerical Analysis

Homework #10 - 2006.11.15

Due Date - 2006.11.20

Solutions

---

1. Consider the initial value problem

$$\begin{cases} y' = 2(t+1)y \\ y(0) = 1 \\ y \in [0,1]\,. \end{cases}$$

a) What is the Lipschitz constant L for the rectangle $S = [0,1] \times \mathbb{R}$?

$$|f(t,y_1) - f(t,y_2)| = |2(t+1)y_1 - 2(t+1)y_2| = |2(t+1)|\,|y_1 - y_2| \leq 4\,|y_1 - y_2|\,.$$

Thus $L = 4$ for all (t, y_1) and (t, y_2) in S.

b) Solve the IVP by either separation of variables or by the method of integrating factors.

We will perform separation of variables:

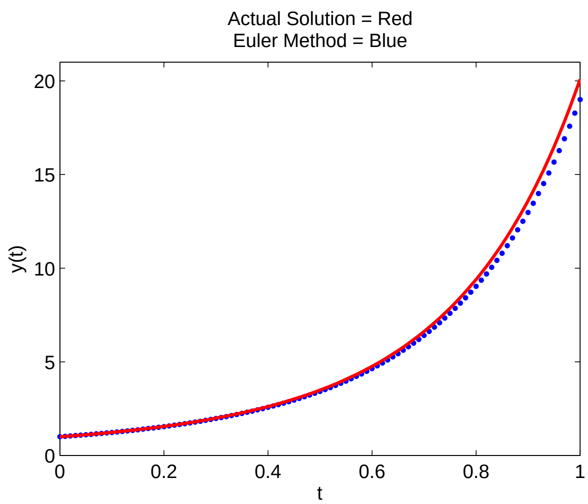
$$\frac{dy}{dt} = 2(t+1)y \Rightarrow \frac{dy}{y} = 2(t+1)dt$$

Integrating over the correct intervals next:

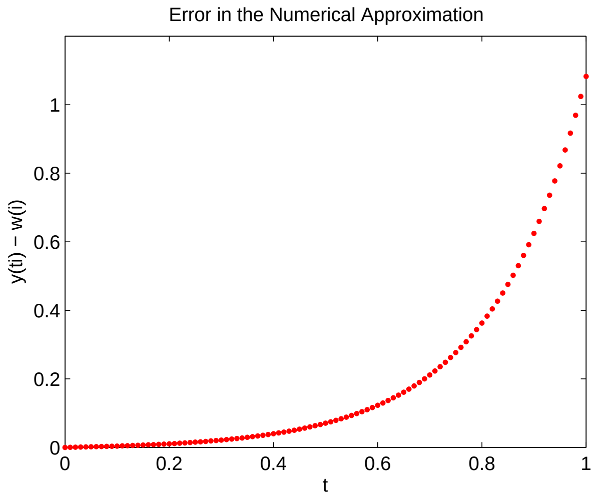
$$\int_1^y \frac{dy}{y} = \int_0^t 2(t+1)dt$$

$$\ln(y) = (t+1)^2 - 1 \Rightarrow y = e^{(t+1)^2-1}.$$

c) Use Euler's Method with $h = 0.01$ to numerically compute the solution to the IVP. Plot both the actual solution and the Euler's Method solution on the same graph.



d) Numerically compute the global error by plotting the difference between the actual solution found in part b) and the Euler's Method solution found in part c).



e) Find the error bound algebraically for Euler's Method applied to the IVP problem. How close does the actual error come to the error bound?

Remember the formula is given by

$$g_i = |w_i - y(t_i)| \leq \frac{Mh}{2L}\left(e^{L(t_i-a)} - 1\right).$$

To bound the error, set $t_i = 1$, the maximum time of interest in this problem, $a = 0$ which is the starting time, $L = 4$ as computed in part a) and $M = 18e^3$, the maximum of $y''(t)$ (computed from the known solution found in part b)) and $h = 0.01$. This gives the maximum global error on the time interval $[0, 1]$ to be $g_{max} \leq 24.22232147$, which is clearly an overestimate.

2. Repeat problem 1 except this time, use the IVP

$$\begin{cases} y' = t^2 y \\ y(0) = 1 \\ y \in [0, 1]. \end{cases}$$

a) What is the Lipschitz constant L for the rectangle $S = [0, 1] \times \mathbb{R}$?

$$|f(t, y_1) - f(t, y_2)| = \left|t^2 y_1 - t^2 y_2\right| = \left|t^2\right| |y_1 - y_2| \leq |y_1 - y_2|,$$

Thus $L = 1$ for all (t, y_1) and (t, y_2) in S.

b) Solve the IVP by either separation of variables or by the method of integrating factors.

We will perform separation of variables:

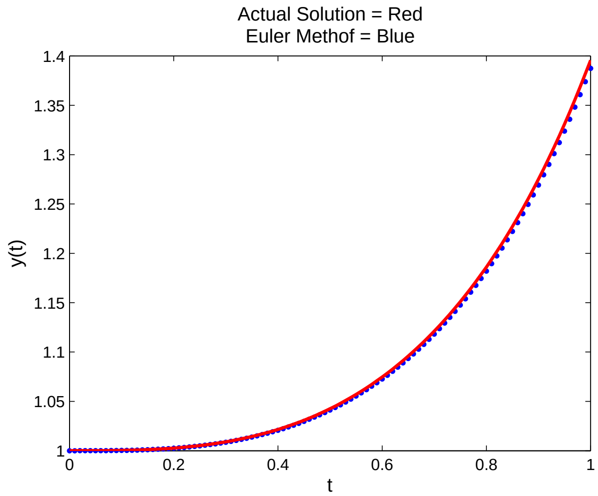
$$\frac{dy}{dt} = t^2 y \Rightarrow \frac{dy}{y} = t^2 dt$$

Integrating over the correct intervals next:

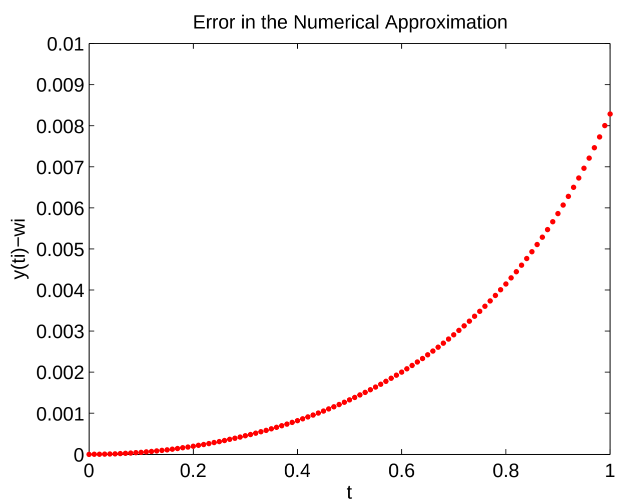
$$\int_1^y \frac{dy}{y} = \int_0^t t^2 dt$$

$$\ln(y) = \frac{1}{3}t^3 \Rightarrow y = e^{\frac{1}{3}t^3}.$$

c) Use Euler's Method with $h = 0.01$ to numerically compute the solution to the IVP. Plot both the actual solution and the Euler's Method solution on the same graph.



d) Numerically compute the global error by plotting the difference between the actual solution found in part b) and the Euler's Method solution found in part c).



e) Find the error bound algebraically for Euler's Method applied to the IVP problem. How close does the actual error come to the error bound?

Once again, the formula is given by

$$g_i = |w_i - y(t_i)| \leq \frac{Mh}{2L}\left(e^{L(t_i - a)} - 1\right).$$

To bound the error, set $t_i = 1$, the maximum time of interest in this problem, $a = 0$ which is the starting time, $L = 1$ as computed in part a) and $M = 3e^{\frac{1}{3}}$, the maximum of $y''(t)$ (computed from the known solution found in part b)) and $h = 0.01$. This gives the maximum global error on the time interval $[0, 1]$ to be $g_{max} \leq 0.03597083204$, which is about 4 times the actual maximum global error.