CS 3513 - Numerical Analysis

Homework #11 - 2006.11.29

Due Date - 2006.12.06

Solutions

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1. Consider the initial value problem

$$\begin{cases} y' = 6y - 6y^2 \\ y(0) = \frac{1}{2} \\ y \in [0, 20]\,. \end{cases}$$

a) If $y(0) = \frac{1}{2}$, what values must $y(t)$ lie between for all $t \geq 0$?

Since the RHS of the IVP can be written as $6y(1-y)$, this states that $y = 0$ and $y = 1$ are fixed points. So with $y(0) = \frac{1}{2}$, the true solution $y(t)$ to the IVP must satisfy $0 \leq y(t) \leq 1$ for all $t \geq 0$.

b) Solve the IVP by separation of variables. Hint: remember partial fractions!

$$\frac{dy}{y - y^2} = 6dt$$

where

$$\frac{1}{y - y^2} = \frac{1}{y} + \frac{1}{1-y}$$

so

$$\int_{\frac{1}{2}}^{y} \frac{1}{y} + \frac{1}{1-y} dy = \int_0^t 6dt.$$

Integrating yields

$$\ln(y) - \ln\left(\frac{1}{2}\right) - \ln(1-y) + \ln\left(\frac{1}{2}\right) = 6t,$$

which simplifies to

$$\frac{y}{1-y} = 6t$$

and solving for y gives

$$y(t) = \frac{e^{6t}}{1 + e^{6t}} = \frac{1}{1 + e^{-6t}}$$

c) Find the formula for w_{i+1} in the Backward Euler Method explicitly, and be sure to use part a) to help finalize your answer.

The formula stated implicitly is given by

$$w_{i+1} = w_i + h\left(6w_{i+1} - 6w_{i+1}^2\right)$$

which can be rewritten as

$$6hw_{i+1}^2 + (1 - 6h)w_{i+1} - w_i = 0.$$

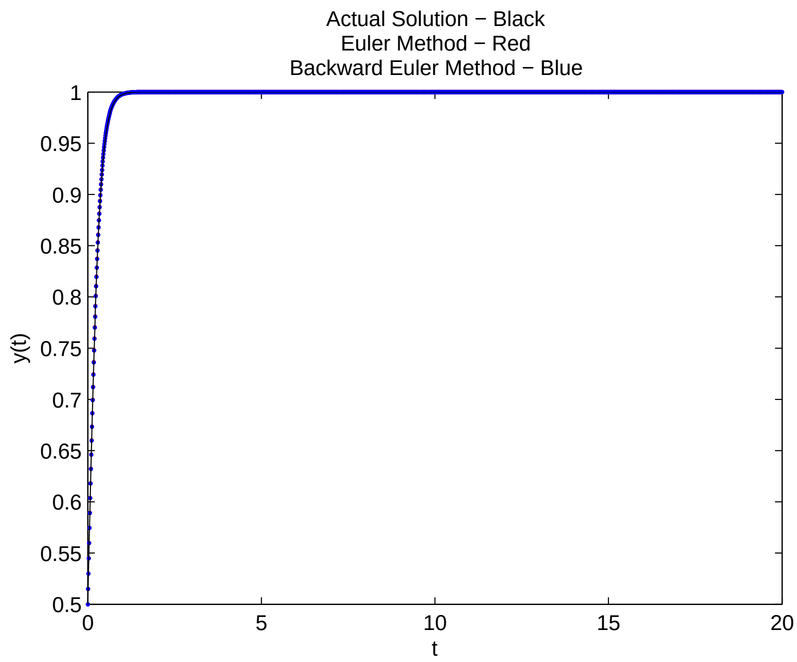
Using the quadratic formula gives

$$\frac{(6h-1) \pm \sqrt{(1-6h)^2 + 24hw_i}}{12h}.$$

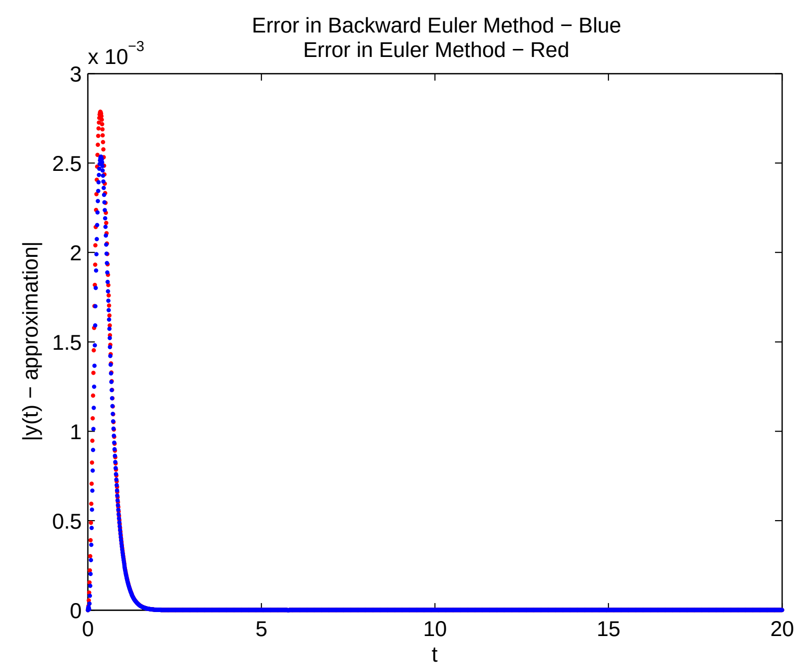
Using part a), we must use the positive portion, which gives

$$\frac{(6h-1) + \sqrt{(1-6h)^2 + 24hw_i}}{12h}.$$

d) Plot the actual solution $Y(t)$, the Backward Euler Method solution $w(t)$, and the Euler Method solution $z(t)$ all on the same graph.



e) Plot $|Y(t) - z(t)|$ and $|Y(t) - w(t)|$ on the same graph.



2. Consider the initial value problem

$$\begin{cases} x' = \frac{1}{2}\frac{1-t}{t}x \\ x(1) = 1 \\ x \in [1, 15] . \end{cases}$$

a) Solve the IVP by separation of variables.
Separating variables gives:

$$\frac{dx}{x} = \frac{1}{2}\frac{1-t}{t}dt$$

and integrating:

$$\int_1^x \frac{dx}{x} = \int_t^t \frac{1}{2}\frac{1-t}{t}dt$$

gives

$$\ln(x) = \frac{1}{2}\left(\ln(t) - t + 1\right).$$

Solving for x gives

$$x(t) = \sqrt{t}e^{-\frac{t}{2}+\frac{1}{2}}.$$

b) Find the formula for w_{i+1} in the Backward Euler Method explicitly, and be sure to use part a) to help finalize your answer.

The formula stated implicitly is given by

$$w_{i+1} = w_i + hf\left(t_{i+1}, w_{i+1}\right) = w_i + h\frac{1}{2}\left(\frac{1-(t+h)}{t+h}\right)w_{i+1}$$

Moving all terms with w_{i+1} to one side gives

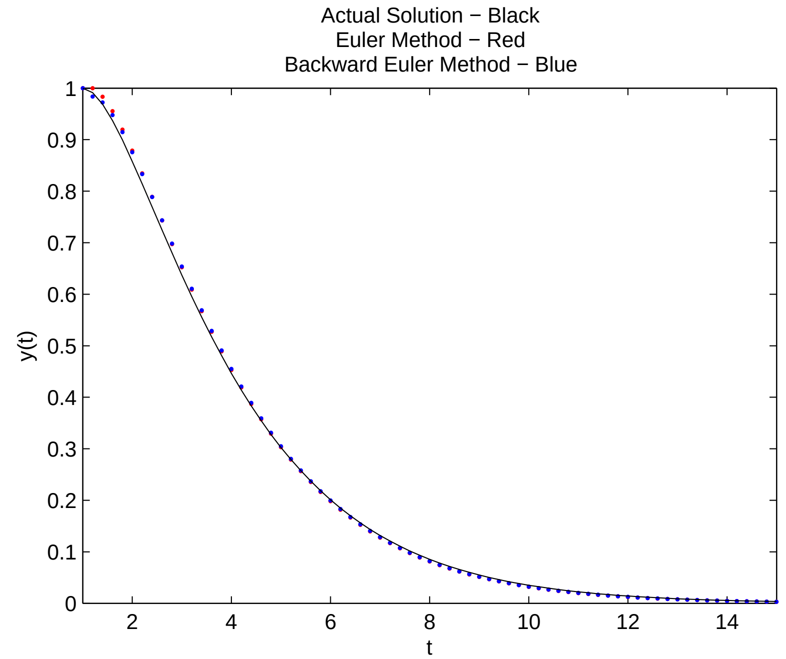
$$w_{i+1}\left(1 - \frac{h}{2}\left(\frac{1-(t+h)}{t+h}\right)\right) = w_i.$$

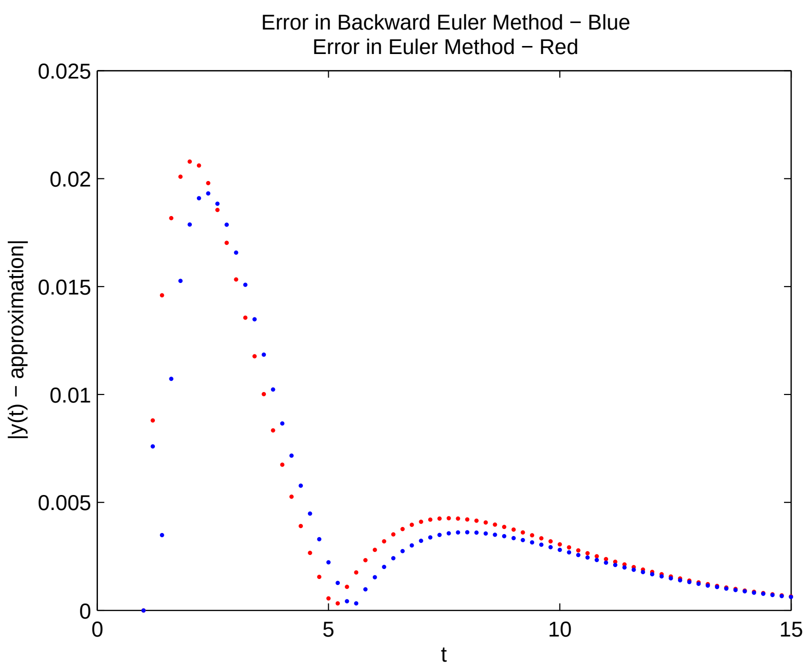
Dividing through gives

$$w_{i+1} = \frac{w_i}{\left(1 - \frac{h}{2}\left(\frac{1-(t+h)}{t+h}\right)\right)}.$$

c) Plot the actual solution $Y(t)$, the Backward Euler Method solution $w(t)$, and the Euler Method solution $z(t)$ all on the same graph.

d) Plot $|Y(t) - z(t)|$ and $|Y(t) - w(t)|$ on the same graph.


Actual Solution − Black
Euler Method − Red
Backward Euler Method − Blue
y(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
4
6
8
10
12
14


Error in Backward Euler Method − Blue
Error in Euler Method − Red
|y(t) − approximation|
t
0
0.005
0.01
0.015
0.02
0.025
0
5
10
15