CS 3513 - Numerical Analysis

Homework #1 - 2006.08.23

Due Date - 2006.08.30

Solutions

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1. Use the Intermediate Value Theorem to prove that $f(c) = 0$ for $c \in (0, 1)$.

a) $f(x) = x^3 - 4x + 1$

Clearly $f(x)$ is continuous. Notice that $-2 = f(1) < 0 < f(0) = 1$. Therefore the IVT applies. A root exists on $(0, 1)$.

b) $f(x) = 5\cos(\pi x) - 4$

Clearly $f(x)$ is continuous. Notice that $-9 = f(1) < 0 < f(0) = 1$. Therefore the IVT applies. A root exists on $(0, 1)$.

c) $f(x) = 8x^4 - 8x^2 + 1$

Clearly $f(x)$ is continuous. Notice that $-\frac{1}{2} = f\left(\frac{1}{2}\right) < 0 < f(0) = 1$. Therefore the IVT applies. A root exists on $(0, 1)$.

2. Find c satisfying the Mean Value Theorem for $f(x)$ on the interval $[0, 1]$.

a) $f(x) = e^x$

Notice that $f'(x) = e^x$, and the MVT implies that there is a $c \in (0, 1)$ such that

$$e^c = \frac{e - 1}{1 - 0} = e - 1$$

which implies that $c = \ln(e - 1) \in (0, 1)$.

b) $f(x) = x^2$

Notice that $f'(x) = 2x$, and the MVT implies that there is a $c \in (0, 1)$ such that

$$2c = \frac{1 - 0}{1 - 0} = 1$$

which implies that $c = \frac{1}{2} \in (0, 1)$.

c) $f(x) = \frac{1}{x+1}$

Here $f'(x) = \frac{-1}{(x+1)^2}$, and the MVT implies that there is a $c \in (0, 1)$ such that

$$\frac{-1}{(c + 1)^2} = \frac{\frac{1}{2} - 1}{1 - 0} = -\frac{1}{2}$$

which implies that $c = -1 \pm \sqrt{2}$. Notice that $-1 \pm \sqrt{2}$ is the solution which lies on $(0, 1)$.

3. Find c satisfying the Mean Value Theorem for Integrals with $f(x)$, $g(x)$ in the interval $(0, 1)$.

a) $f(x) = x$, $g(x) = x$

Here we have

$$\int_0^1 x^2 \; dx = c \int_0^1 x \; dx$$

which gives

$$c = \frac{\frac{1}{3}x^3 \mid_0^1}{\frac{1}{2}x^2 \mid_0^1} = \frac{2}{3} \in (0, 1).$$

b) $f(x) = x^2$, $g(x) = x$

Here we have

$$\int_0^1 x^3 \; dx = c^2 \int_0^1 x \; dx$$

which gives

$$c^2 = \frac{\frac{1}{4}x^4 \mid_0^1}{\frac{1}{2}x^2 \mid_0^1} = \pm\frac{1}{2}.$$

Notice that the positive root, $c = \frac{1}{\sqrt{2}} \in (0,1)$.

c) $f(x) = x$, $g(x) = e^x$

Once again, using the formula yields

$$\int_0^1 xe^x \, dx = c \int_0^1 e^x \, dx$$

which gives

$$c = \frac{xe^x - e^x \mid_0^1}{e^x \mid_0^1} = \frac{1}{e-1} \in (0,1).$$

Notice that the positive root, $c = \frac{1}{\sqrt{2}} \in (0,1)$.

4. Find the Taylor polynomial of degree 2 about the point $x = 0$ for the following functions:

a) $f(x) = e^{x^2}$

The formula is $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$. In this case, $f'(x) = 2xe^{x^2}$ and $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$. This $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 2$. Plugging this into the formula for $p_2(x)$ gives

$$p_2(x) = 1 + x^2.$$

b) $f(x) = \cos(5x)$

Simlar to part a), we just compute the derivatives and evaluate at $x = 0$. This gives $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -25$. Therefore

$$p_2(x) = 1 - \frac{25}{2}x^2.$$

c) $f(x) = \frac{1}{x+1}$

Once again, $f(0) = 1$, $f'(0) = -1$ and $f''(0) = 2$, therefore

$$p_2(x) = 1 - x + x^2.$$

5. Find the Taylor polynomial of degree 5 about the point $x = 0$ for the following functions:

a) $f(x) = e^{x^2}$

Similar work can be done here to arrive at

$$p_5(x) = 1 + x^2 + \frac{1}{2}x^4.$$

b) $f(x) = \cos(2x)$

Same here,

$$p_5(x) = 1 - 2x^2 + \frac{2}{3}x^4.$$

c) $f(x) = \ln(1 + x)$

And again:

$$p_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

d) $f(x) = \sin^2(x)$

$$p_5(x) = x^2 - \frac{1}{3}x^4$$

6. a) Find the Taylor polynomial of degree 4 for $f(x) = x^{-2}$ about the point $x = 1$.

The formula changes a little bit here:

$$p_4(x) = f(0) + f'(0)(x-1) + \frac{1}{2}f''(0)(x-1)^2 + \frac{1}{6}f'''(0)(x-1)^3 + \frac{1}{24}f^{(v)}(0)(x-1)^4$$

Taking derivatives and plugging into the formula gives

$$p_4(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4$$

b) Use the result of a) to approximate $f(0.9)$ and $f(1.1)$.

$$f(0.9) = 1.2345 \text{ and } f(1.1) = 0.8265$$

c) Use the Taylor remainder to find an error formula for the Taylor polynomial. Give error bounds for each of the two approximations made in part b). Which of the two approximations in part b) do you expect to be closer to the correct value?

The error bound is given by

$$6\left|x-1\right|^5 c^{-7} = 6\frac{\left|x-1\right|^5}{c^7}$$

where c must be found to maximize error to give a proper bound. Since to maximize we have to minimize the denominator, we can choose a value of $c = 0.9$. So for $c = 0.9$, the error is bounded by .0001254450949 and for $x = 1.1$ the error bound is the same.

One would expect the approximation to be better for for $x > 1$ since as $x \to 0$, the derivative increases greatly.

d) Use a calculator to compare the actual error in each case with your error bound from part c).

The true value of $\frac{1}{0.9^2}$ is approximately 1.234567901. The approximated value is 1.2345, so the difference is approximately 0.000067901, which is smaller than the error bound.

The true value of $\frac{1}{1.1^2}$ is approximately 0.8264462810. The approximated value is 0.8265, so the difference is approximately 0.0000537190, which is smaller than the error bound once again.

7. a) Find the degree 5 Taylor polynomial $P_5(x)$ centered at $x = 0$ for $f(x) = \cos(x)$.

The Taylor formula for $\cos(x)$ is well known.

$$P_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

b) Find an upper bound for the error in approximating $f(x) = \cos(x)$ for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ by $P_5(x)$.

The error term in the approximation is

$$\frac{1}{720}\left|\cos(c)\right|\left|x\right|^6$$

and since $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, to maximize, one chooses $c = 0$. So the error bound is

$$\frac{1}{720}\left|x\right|^6.$$

To place an upper bound on the error for any $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, we substitute $|x| = \frac{\pi}{4}$ to get a maximum error of .0003259918872.

8. A common approximation for $\sqrt{1+x}$ is $1 + \frac{1}{2}x$, when x is small. Use the degree 1 Taylor polynomial of $f(x) = \sqrt{1+x}$ with remainder to determine a formula of form $\sqrt{1+x} = 1 + \frac{1}{2}x \pm E$. Evaluate E for the case of approximating $\sqrt{1.02}$. Use a calculator to compare the actual error to your error bound E.

The first order approximation is $P_1(x) = 1 + \frac{1}{2}x + E$ where

$$E = \frac{1}{2}\left|\frac{1}{4\left(1+c\right)^{\frac{3}{2}}}\right|x^2 = \frac{1}{8}\left|\frac{1}{\left(1+c\right)^{\frac{3}{2}}}\right|x^2.$$

To maximize E, one must chose $c = 0$, thus

$$E = \frac{1}{8}\left|x\right|^2.$$

To approximate $\sqrt{1.02}$, we have $\sqrt{1.02} \approx 1 + \frac{1}{2}0.02 = 1.01$. The error is bounded by $\frac{1}{8}\left|0.02\right|^2 = 0.00005$. The actual value of $\sqrt{1.02}$ is approximately 1.009950494, and the error is .000049506 which is barely under the error bound!