CS 3513 - Numerical Analysis

Homework #2 - 2006.09.05

Due Date - 2006.09.15

Name: ____________________________

This HW will discuss the FPI scheme for computing the square root of a positive integer. We will start with $\sqrt{2}$.

1. Given $x_0 > 0$, show:

a) If $0 < x_0 < \sqrt{2}$, then $\frac{2}{x_0} > \sqrt{2}$.

b) If $x_0 > \sqrt{2}$, then $0 < \frac{2}{x_0} < \sqrt{2}$.

c) If $x_0 < \frac{2}{x_0}$, and $x_0 > 0$, then what values can x_0 take on? What values can $\frac{2}{x_0}$ take on?

d) If $x_0 > \frac{2}{x_0}$, and $x_0 > 0$, then what values can x_0 take on? What values can $\frac{2}{x_0}$ take on?

2. Let

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right).$$

Answer the following:

a) Argue that if $x_0 > 0$, then $x_i > 0$ for all i.

b) If $\frac{2}{x_i} < x_i$, show that $\frac{2}{x_i} < x_{i+1}$.

c) Show that part b) implies that $\frac{2}{x_{i+1}} < x_i$

d) Is it necessarily going to be true that $\frac{2}{x_{i+1}} < x_{i+1}$ or vice versa? To determine this, consider the function involved in the FPI scheme. Here we have $x_{i+1} = g(x_{i+1})$ with

$$g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right).$$

What does this function look like? What properties of $g(x)$ can you use to help answer this question?

e) Use parts a) through d) to show that $\frac{2}{x_i} < \frac{2}{x_{i+1}} < x_{i+1} < x_i$.

3. Now that we know something about the behaviour of the x_i's, we will now finish off the problem.

a) On what finite interval must all x_i's lie for $i > 0$?

b) Since x_i is finite for all i, we can consider taking a limit in the equation

$$x_{i+1} = g(x_i).$$

If a fixed point does occur and all the sequence of x_i's are bounded, then we could take limits:

$$\lim_{i \to \infty} x_{i+1} = \lim_{i \to \infty} g(x_i).$$

If there is indeed a fixed point, call it $x = r$, use the above limiting form to find it.

4. Explain how the above method can be modified to find the square root of another positive integer, say 11 instead of 2.