# CS 3513 - Numerical Analysis

## Homework #2 - 2006.09.05

## Due Date - 2006.09.15

## Solutions

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This HW will discuss the FPI scheme for computing the square root of a positive integer. We will start with $\sqrt{2}$.
1. Given $x_0 > 0$, show:

a) If $0 < x_0 < \sqrt{2}$, then $\frac{2}{x_0} > \sqrt{2}$.

Notice that

$$0 < x_0 < \sqrt{2} \Leftrightarrow \frac{1}{x_0} > \frac{1}{\sqrt{2}} \Leftrightarrow \frac{2}{x_0} > \frac{2}{\sqrt{2}}$$

but $\frac{2}{\sqrt{2}} = \sqrt{2}$, therefore $\frac{2}{x_0} > \sqrt{2}$.

b) If $x_0 > \sqrt{2}$, then $0 < \frac{2}{x_0} < \sqrt{2}$.

Changing the inequality signs in part a) gives the desired result here.

c) If $x_0 < \frac{2}{x_0}$, and $x_0 > 0$, then what values can $x_0$ take on? What values can $\frac{2}{x_0}$ take on?

We simply solve the inequality for $x_0$. This gives $x_0^2 < 2$ or $0 < x_0 < \sqrt{2}$. Using part a), this gives that $\sqrt{2} < \frac{2}{x_0}$.

d) If $x_0 > \frac{2}{x_0}$, and $x_0 > 0$, then what values can $x_0$ take on? What values can $\frac{2}{x_0}$ take on?

Here, solving for $x_0$ gives $x_0^2 > 2$, which means $x_0 > \sqrt{2}$. Thus $0 < \frac{2}{x_0} < \sqrt{2}$ from part b).

2. Let

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right).$$

Answer the following:

a) Argue that if $x_0 > 0$, then $x_i > 0$ for all $i$.

Since $x_{i+1}$ is simply the average of $x_i$ and $\frac{2}{x_i}$, then as long as $x_i$ is positive, $x_{i+1}$ is positive. This is true if one assumes that $x_0 > 0$.

b) If $\frac{2}{x_i} < x_i$, show that $\frac{2}{x_i} < x_{i+1}$.

Simply start with the inequality $\frac{2}{x_i} < x_i$ and add $\frac{2}{x_i}$ to both sides. This gives $\frac{4}{x_i} < x_i + \frac{2}{x_i}$. Next, divide both sides by 2 to get $\frac{2}{x_i} < x_{i+1}$.

c) Show that part b) implies that $\frac{2}{x_{i+1}} < x_i$

Here we simply use the inequality we proved in part b). If $\frac{2}{x_i} < x_{i+1}$ is true, then we can multiply both sides by $x_i$ and divide both sides by $x_{i+1}$ to get $\frac{2}{x_{i+1}} < x_i$.

d) Is it necessarily going to be true that $\frac{2}{x_{i+1}} < x_{i+1}$ or vice versa? To determine this, consider the function involved in the FPI scheme. Here we have $x_{i+1} = g(x_{i+1})$ with

$$g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right).$$

What does this function look like? What properties of $g(x)$ can you use to help answer this question?

The function $g(x)$ goes to infinity at both $x = 0$ and $x = \infty$. There is only one critical point, it happens to be at $x = \sqrt{2}$ where it obtains its minimum value. The minimum value is $g(\sqrt{2}) = \sqrt{2}$. Thus, the smallest $g(x_i)$ can ever be is $\sqrt{2}$, which it obtains ONLY when $x_i = \sqrt{2}$. So any other value of $x_i$ gives a value greater. This means that $\frac{2}{x_i}$ must be smaller than $\sqrt{2}$.

e) Use parts a) through d) to show that $\frac{2}{x_i} < \frac{2}{x_{i+1}} < x_{i+1} < x_i$.

This follows immediately from what was discussed in parts a) - d).

3. Now that we know something about the behaviour of the $x_i$'s, we will now finish off the problem.

a) On what finite interval must all $x_i$'s lie for $i > 0$?

If $x_0 > \sqrt{2}$, then all $x_i$'s lie on the interval $(\frac{2}{x_i}, x_i)$. If $x_0 < \sqrt{2}$, then the endpoints simply swap.

b) Since $x_i$ is finite for all $i$, we can consider taking a limit in the equation

$$x_{i+1} = g(x_i).$$

If a fixed point does occur and all the sequence of $x_i$'s are bounded, then we could take limits:

$$\lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} g(x_i).$$

If there is indeed a fixed point, call it $x = r$, use the above limiting form to find it.

Taking a limit, then we have

$$\lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} x_i = r$$

so

$$r = g(r).$$

This has the form

$$r = \frac{r + \frac{2}{r}}{2}.$$

Solving for $r$ gives $r^2 = 2$ or $r = \sqrt{2}$. Here we disregard the negative solution since all $x_i$'s are positive.

4. Explain how the above method can be modified to find the square root of another positive integer, say 11 instead of 2.

The only difference in any of the analysis would be to replace $\sqrt{2}$ everywhere with $\sqrt{11}$, and $g(x_i)$ would have to be modified as

$$g(x_i) = \frac{1}{2}\left(x_i + \frac{11}{x_i}\right).$$