

CS 3513 - Numerical Analysis

Homework #3 - 2006.09.13

Due Date - 2006.09.25

Solutions

1. Consider the polynomial $P(x) = (x-1)^4(2x+1)^2(4x+7)x$ which has roots at $x = 1$, $x = -\frac{1}{2}$, $x = -\frac{7}{4}$ and $x = 0$. For each root, which method will converge faster, the Bisection Method or Newton's Method?

Notice that the root at $x = 1$ is a root of order 4, thus the Bisection method will converge faster. The root at $x = -\frac{1}{2}$ is a root of order two, so the rate of convergence will be the same for both methods. Both $x = -\frac{7}{4}$ and $x = 0$ are simple roots, so Newton's Method will converge faster.

2. Prove Newton's Method applied to $f(x) = ax + b$ converges in one step.

Notice that the true root is at $x = -\frac{b}{a}$. Newton's method is given by the formula

$$x_1 = x_0 - \frac{ax_0 + b}{a}.$$

Simplifying the above expression gives $x_1 = -\frac{b}{a}$.

3. Show that Newton's Method applied to the equation $f(x) = x^2 - A$ produces the iteration used to find $\sqrt{2}$ in the previous homework assignment when $A = 2$.

The iteration process for Newton's Method is given by

$$x_{n+1} = x_n - \frac{x_n^2 - A}{2x_n} = \frac{1}{2} \left(x_n + \frac{A}{x_n} \right)$$

When $A = 2$, we get the form of the FPI from the previous homework assignment.

4. Use Newton's Method to produce a quadratically convergent method for calculating the n^{th} root of a positive number A where n is a positive integer. Prove quadratic convergence.

Start with $f(x) = x^n - A$, then Newton's Method yields

$$x_{k+1} = x_k - \frac{x_k^n - A}{nx_k^{n-1}}.$$

Upon simplification, one has

$$x_{k+1} = \frac{(n-1)x_k^n - A}{nx_k^{n-1}} = \frac{1}{n} \left((n-1)x_k + \frac{A}{x_k^{n-1}} \right)$$

Notice that when $n = 2$, one has the FPI from the previous problem.

Furthermore, by waving our hands, if there is a fixed point r to

$$x_{k+1} = \frac{(n-1)x_k^n - A}{nx_k^{n-1}} = \frac{1}{n} \left((n-1)x_k + \frac{A}{x_k^{n-1}} \right)$$

then one has

$$r = \frac{1}{n} \left((n-1)r + \frac{A}{r^{n-1}} \right).$$

Solving for r gives $r = A^{\frac{1}{n}}$.

To prove quadratic convergence, we check the constant of convergence, M . Notice that

$$f' \left(A^{\frac{1}{n}} \right) = nA \cdot A^{-\frac{1}{n}} \neq 0$$

which implies quadratic convergence and

$$f'' \left(A^{\frac{1}{n}} \right) = n(n-1)A \cdot A^{-\frac{2}{n}}$$

and thus

$$M = \frac{n-1}{2} \left(\frac{1}{A} \right)^{\frac{1}{n}}.$$

Therefore,

$$e_{k+1} \approx \frac{n-1}{2} \left(\frac{1}{A} \right)^{\frac{1}{n}} e_k^2.$$

Computer Problems:

Each of the following equations has one root. Use a) Bisection Method, b) Newton's Method and c) Secant Method to approximate the root to eight correct decimal places. In each case, state your initial condition(s), the number of iterations required to acquire accuracy to eight decimal places, along with the output at each step.

1. $f(x) = x^5 + x - 1$

Answers will vary.

2. $g(x) = \sin(x) - 6x - 5$

Answers will vary.

3. $h(x) = e^x + \sin(x) - 4$

Answers will vary.