CS 3513 - Numerical Analysis

Homework #7 - 2006.10.25

Due Date - 2006.10.30

Solutions

Consider the best fit line $y = mx + b$ for the three points (x_1, y_1), (x_2, y_2) and (x_2, y_3). (Notice that the last two points have the same $x-$coordinate.)

1. Find the equation of the line which goes through the points (x_1, y_1) and the point half-way between (x_2, y_2) and (x_2, y_3), and write it in slope-intercept form.

The second point is given to be $(x_2, \frac{y_2+y_3}{2})$. Thus

$$m = \frac{\frac{y_2+y_3}{2} - y_1}{x_2 - x_1} = \frac{2y_1 - y_2 - y_3}{2(x_1 - x_2)}$$

and using the point slope form, one has

$$y - y_1 = \frac{2y_1 - y_2 - y_3}{2(x_1 - x_2)}(x - x_1)$$

which can be rewritten as

$$y = \frac{2y_1 - y_2 - y_3}{2(x_1 - x_2)}x + \frac{x_1y_2 + x_1y_3 - 2x_2y_1}{2(x_1 - x_2)},$$

so

$$b = \frac{x_1y_2 + x_1y_3 - 2x_2y_1}{2(x_1 - x_2)}.$$

The equation of the line which goes through the two points is now given by:

$$y = \frac{2y_1 - y_2 - y_3}{2(x_1 - x_2)}x + \frac{x_1y_2 + x_1y_3 - 2x_2y_1}{2(x_1 - x_2)}.$$

2. Compute the values of m and b by the method of least squares.

We start with the system of equations found by using the form $y = mx+b$ along with the three points in question.

$$\begin{cases} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ y_3 = mx_2 + b \end{cases}$$

This yields the matrix system $A\overrightarrow{x} = \overrightarrow{c}$ given by

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Notice that $A^\top A$ is given by

$$A^\top A = \begin{bmatrix} x_1^2 + 2x_2^2 & x_1 + 2x_2 \\ x_1 + 2x_2 & 3 \end{bmatrix}$$

and that $A^\top \overrightarrow{c}$ is

$$\begin{bmatrix} x_1y_1 + x_2(y_2 + y_3) \\ y_1 + y_2 + y_3 \end{bmatrix}$$

Multiplying the equations out gives

$$\begin{cases}(x_1^2+2x_2^2)m+(x_1+2x_2)b = x_1y_1+x_2(y_2+y_3)\\(x_1+2x_2)m+3b = y_1+y_2+y_3.\end{cases}$$

Solving the second equation for b gives

$$b = \frac{y_1+y_2+y_3-(x_1+2x_2)m}{3}$$

and plugging this into the first equation yields an equation solvable in terms of m:

$$(x_1^2+2x_2^2)m+(x_1+2x_2)\frac{y_1+y_2+y_3-(x_1+2x_2)m}{3} = x_1y_1+x_2(y_2+y_3).$$

After a few algebraic manipulations including canceling and factoring, one arrives at

$$2(x_1-x_2)^2m = (2y_1-y_2-y_3)(x_1-x_2)$$

which, after one more simple step, gives the desired result of

$$m = \frac{2y_1-y_2-y_3}{2(x_1-x_2)}.$$

Once m has been found, we substitute this into the second equation after we had solved for b:

$$b = \frac{y_1+y_2+y_3-(x_1+2x_2)m}{3} = \frac{1}{3}\left[y_1+y_2+y_3-\frac{2y_1-y_2-y_3}{2(x_1-x_2)}\right].$$

After finding a common denominator, expanding the resulting numerator and canceling/combing like terms, one gets:

$$b = \frac{x_1y_2+x_1y_3-2x_2y_1}{2(x_1-x_2)}.$$

This gives

$$y = \frac{2y_1-y_2-y_3}{2(x_1-x_2)}x+\frac{x_1y_2+x_1y_3-2x_2y_1}{2(x_1-x_2)}.$$

3. Show that the lines from the above two problems are the same. (I.E. show they go through the two points described in problem 1 or show that their formulas are identical).

This follows immediately from the fact that the lines in problems 1 and 2 are the same, and the line in problem 1 goes through the points described in the question.