# CS 3513 - Numerical Analysis
## Homework #8 - 2006.11.01
## Due Date - 2006.11.08
## Solutions

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Use the $QR$-factorization to find the least squares solutions and 2-norm error of the following inconsistent systems. Be sure to show ALL work.

1.

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ -5 \\ 15 \\ 0 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix}.$$

Performing $QR$-factorization on the three column vectors found in $A$ gives

$$Q \approx \begin{bmatrix} .4803844614 & -.2696900347 & .4057324979 \\ .6405126154 & .5493685892 & -.2236476696 \\ -.4803844614 & .6592423071 & -.03100719902 \\ .1601281538 & .4295063516 & .6913945655 \\ -.3202563076 & -.07990815844 & .5535114890 \end{bmatrix}, \quad R \approx \begin{bmatrix} 6.244997998 & -.6405126154 & .3202563076 \\ 0 & 2.567049590 & 2.027669520 \\ 0 & 0 & 5.897965091 \end{bmatrix}.$$

Tacking on two more vectors to span $\mathbb{R}^5$ will fill out the rest of $Q$. So, choose $v_4 = [1,0,0,0,0]^\top$ and $v_5 = [0,1,0,0,0]^\top$. Adding these, and performing the Gram-Schmidt Orthonormalization process with the already found vectors $\overrightarrow{q_1}$, $\overrightarrow{q_2}$, and $\overrightarrow{q_3}$ along with $\overrightarrow{v_4}$ and $\overrightarrow{v_5}$ gives

$$Q \approx \begin{bmatrix} .4803844614 & -.2696900347 & .4057324979 & .7293004832 & 0 \\ .6405126154 & .5493685892 & -.2236476696 & -.09432593001 & .4785625163 \\ -.4803844614 & .6592423071 & -.03100719902 & .577458742 & -.01450189444 \\ .1601281538 & .4295063516 & .6913945655 & -.3312910712 & -.4495587274 \\ -.3202563076 & -.07990815844 & .5535114890 & -.1265347842 & .7540985103 \end{bmatrix}$$

and the new $R$ is given by

$$R \approx \begin{bmatrix} 6.244997998 & -.6405126154 & .3202563076 \\ 0 & 2.567049590 & 2.027669520 \\ 0 & 0 & 5.897965091 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next notice that $Q^\top \overrightarrow{b}$ is given by

$$Q^\top \overrightarrow{b} \approx \begin{bmatrix} 16.01281539 \\ 5.943169283 \\ 12.34680276 \\ -1.506914247 \\ -1.885246276 \end{bmatrix}$$

Our new system is given by

$$\begin{bmatrix} 6.244997998 & -.6405126154 & .3202563076 \\ 0 & 2.567049590 & 2.027669520 \\ 0 & 0 & 5.897965091 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16.01281539 \\ 5.943169283 \\ 12.34680276 \\ -1.506914247 \\ -1.885246276 \end{bmatrix}$$

which has solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \begin{bmatrix} 2.524608502 \\ .661633113 \\ 2.093400447 \end{bmatrix}$$

and 2-norm error $\| e \|_2 = \sqrt{-1.506914247^2 + -1.885246276^2} = 1.132856113$.

2.

$$\begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$

Here

$$A = \begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix}.$$

Performing $QR$-factorization on the three column vectors found in $A$ gives

$$Q \approx \begin{bmatrix} .7184212083 & .2115037440 & .2258696008 & .6085938030 \\ -.3592106041 & .7684636029 & .5228118435 & -.05163137633 \\ .1796053021 & .5992606078 & -.7687752802 & -.1320450358 \\ .1796053021 & -.05640099837 & .05246723839 & -.4071178209 \\ .5388159061 & .04935087358 & .2861511087 & -.6661583693 \end{bmatrix}$$

$$R \approx \begin{bmatrix} 5.567764363 & 1.436842416 & 3.592106041 & -1.257237114 \\ 0 & 4.575530994 & -2.439343180 & 1.924684070 \\ 0 & 0 & 4.140818645 & -1.639508173 \\ 0 & 0 & 0 & 1.423713111 \end{bmatrix}.$$

Tacking on one more vector to span $\mathbb{R}^5$ will fill out the rest of $Q$. So, choose $v_5 = [1, 0, 0, 0, 0]^\top$. Performing the Gram-Schmidt Orthonormalization process with the already found vectors $\overrightarrow{q_1}, \overrightarrow{q_2}, \overrightarrow{q_3}$, and $\overrightarrow{q_4}$ along with $\overrightarrow{v_5}$ gives

$$Q \approx \begin{bmatrix} .7184212083 & .2115037440 & .2258696008 & .6085938030 & .1331677165 \\ -.3592106041 & .7684636029 & .5228118435 & -.05163137633 & .06658385824 \\ .1796053021 & .5992606078 & -.7687752802 & -.1320450358 & -.01331677165 \\ .1796053021 & -.05640099837 & .05246723839 & -.4071178209 & .8922237005 \\ .5388159061 & .04935087358 & .2861511087 & -.6661583693 & -.4261366928 \end{bmatrix}$$

and the new $R$ is given by

$$R \approx \begin{bmatrix} 5.567764363 & 1.436842416 & 3.592106041 & -1.257237114 \\ 0 & 4.575530994 & -2.439343180 & 1.924684070 \\ 0 & 0 & 4.140818645 & -1.639508173 \\ 0 & 0 & 0 & 1.423713111 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next notice that $Q^\top \overrightarrow{b}$ is given by

$$Q^\top \overrightarrow{b} \approx \begin{bmatrix} 10.23750222 \\ 3.560313024 \\ 2.151900992 \\ 2.491056112 \\ -.825639842 \end{bmatrix}.$$

Our new system is given by

$$\begin{bmatrix} 5.567764363 & 1.436842416 & 3.592106041 & -1.257237114 \\ 0 & 4.575530994 & -2.439343180 & 1.924684070 \\ 0 & 0 & 4.140818645 & -1.639508173 \\ 0 & 0 & 0 & 1.423713111 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10.23750222 \\ 3.560313024 \\ 2.151900992 \\ 2.491056112 \\ -.825639842 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 1.273896082 \\ .6885086013 \\ 1.212449017 \\ 1.749689662 \end{bmatrix}$$

and 2-norm error $\| e \|_2 = \sqrt{-.825639842^2} = .825639842$.