Project #1
Fixed Point Iteration (FPI)
The Logistic Map and Bifurcation Diagrams

Consider the function $f(x) = ax(1-x)$, where a is a parameter between 1 and 4. For a fixed point problem, we have the equation

$$x_{n+1} = f(x_n) = ax_n(1-x_n).$$

Now if we assume that $x_0 \in [0,1]$, the questions becomes: What is the behavior of the fixed points as one varies a from 1 to 4? First you should look over the next two theorems which might prove useful when attempting to answer some of the questions for this project.

Theorem 1: If f is continuously differentiable and $|f'(x)| \leq B < 1$ on an interval $[a,b]$ containing the fixed point r, then FPI converges to r from any initial guess in $[a,b]$.

Theorem 2: A continuously differentiable function $f(x)$ satisfying $|f'(x)| < 1$ on a closed interval cannot have two fixed points on that interval.

Questions:

1. For any given a on the interval $[1,4]$, there are two fixed points. One is obviously $x_0 = 0$, the other (call it x_a) depends on a. What is the formula for x_a?

2. Clearly we have two fixed points (namely x_0 and x_a) for EACH value of a. Why does this not contradict Theorem 2?

3. What is $f'(0)$? What does that imply about the map?

4. Consider the function $g(x) = ax(1-x)$. What is the maximum value of this function for $x \in [0,1]$?

5. For values of $a \in [1,4]$, what does problem 4 imply about the maximum value of $g(x)$, and how does this relate to the map $x_{n+1} = ax_n(1-x_n)$?

6. For what values of a is $|f'(x_a)| < 1$? Remember that $a \in [1,4]$, what does this imply about the values of a in your answer? What about those values not in your answer?

Matlab:

Discreetize the values of a on the interval $[1,4]$ so that the distance between any two values of a is sufficiently small enough to discern behaviors on a small scale. For each value of a, start with an arbitrary initial condition $x_0 \in (0,1)$ and perform a FPI on the iterated map $x_{n+1} = ax_n(1-x_n)$. It might help to use the same initial condition x_0 for each value of a.

Next, iterate the map several hundred times, yet store only the last hundred results. Do this for each value of a in your discreetized interval. Afterwards, plot the last hundred iterations for each value of a on a graph (hence the values of a will be on the $x-$axis, and the last hundred iterations for each value of a on the $y-$axis. The resulting graph is sometimes called a bifurcation diagram.

More Questions:

7. Does the graph depict the behaviors found so far in answers to the first 4 problems?

8. If your graph is correct, there will be a splitting from one point to two at a certain value of a, what is this value?

9. For what value of a does the graph split again (thus 4 points are visible)?

10. The split described in problem 6 is sometimes referred to as a period doubling. For values of a between your answers to problem 6 and problem 7, the exhibited behavior of the iterated map is called a '2-cycle' (as opposed to a 1-cycle for values of a previous to your answer to problem 6). These points in the 2-cycle are not fixed points, why?

11. Consider the map $x_{n+1} = f(f(x_n))$. Notice the difference between this map and the map $x_{n+1} = f(x_n)$. Write out explicitly what the map $x_{n+1} = f(f(x_n))$ is in terms of x_n.

12. A fixed point x_p of the equation $x_{n+1} = f(f(x_n))$ must satisfy $x_p = f(f(x_p))$. Substitute $a = 3$ into the expression found in problem 11 and solve for the fixed point(s). Does this agree with your bifurcation diagram?

13. Try this again but let $a = 3.1$. You will have to find the roots numerically. You should have three solutions (besides the zero solution). It may seem like this does not agree with your bifurcation diagram, however it does. Can you explain why? Hint: remember the zero solutions!

Final Remarks:

Please answer each question to the fullest of your abilities and show all your work. When writing up your solutions, be sure to be neat, orderly and thorough. I will also expect a copy of both your code and the results. Emailing is the preferred method of communication.