Math 2013 - Introduction to Discrete Mathematics

Homework #3 - 2005.09.14

Due Date - 2005.09.19

Solutions

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1. Show that if n is an odd integer, then

$$\lceil \frac{n^2}{4} \rceil = \frac{n^2+3}{4}$$

Since n is odd, then $n = 2k+1$, for $k \in \mathbb{Z}$. We thus have

$$\lceil \frac{n^2}{4} \rceil = \lceil \frac{4k^2+4k+1}{4} \rceil = \lceil k^2 + k + \frac{1}{4} \rceil$$

Now notice that one has the following (why??):

$$\lceil k^2 + k + \frac{1}{4} \rceil = \lceil k^2 + k + 1 \rceil$$

$$= k^2 + k + 1 = \frac{4k^2+4k+4}{4} = \frac{4k^2+4k+1+3}{4} = \frac{(2k+1)^2+3}{4} = \frac{n^2+3}{4}$$

2. Let $max\{x, y\}$ denote the maximum of x and y and $min\{x, y\}$ denote the minimum of x and y. Prove the following two statements:

a)$max\{x, y\} + min\{x, y\} = x + y$

If $max\{x, y\} = x$ and $min\{x, y\} = y$, then $max\{x, y\} + min\{x, y\} = x + y$.
If $max\{x, y\} = y$ and $min\{x, y\} = x$, then $max\{x, y\} + min\{x, y\} = x + y$.
If $x = y$, then $max\{x, y\} = min\{x, y\} = x = y$ and $max\{x, y\} + min\{x, y\} = x + x = y + y = x + y$.

b)$max\{x, y\} - min\{x, y\} = |x - y|$

If $max\{x, y\} = x$ and $min\{x, y\} = y$, then $max\{x, y\} - min\{x, y\} = x - y = |x - y|$.
If $max\{x, y\} = y$ and $min\{x, y\} = x$, then $max\{x, y\} - min\{x, y\} = y - x = -(x - y) = |x - y|$.
If $x = y$, then $max\{x, y\} = min\{x, y\} = x = y$ and $max\{x, y\} - min\{x, y\} = x - x = y - y = 0 = |x - y|$.

3. Prove that a bijection exists between any two closed intervals $[a, b]$ and $[c, d]$ where $a < b$ and $c < d$. *Hint: find a formula that works!*

Notice that a linear function, which passes through the points (a, c) and (b, d). The line is given by

$$p(t) = \frac{c-d}{a-b}(t-a) + c, \quad t \in [a, b]$$

The function $p(t)$ is a bijection.

4. Let $\Sigma = \{a, ba, abb, bbb\}$ be an alphabet, and define Σ^* as usual. Is $f(x) = \| x \|$, where $x \in \Sigma^*$ a function? Why or why not?

Notice that the word $abbba \in \Sigma^*$, can be expressed as

$$abbba = a \cdot bbb \cdot a = abb \cdot ba.$$

Thus, $f(abbba)$ is not defined for $abbba$, as it can be either 2 or 3.

5. Compute the domain and range of the following functions.

a) $f(x) = x^2 - 1$

Domain is $\mathbb{R}$, range is $[-1, \infty)$.

b) $g(x) = \frac{1}{x}$

Domain is $(-\infty, 0) \cup (0, \infty)$, range is $(-\infty, 0) \cup (0, \infty)$.

c) $h(x) = (f + g)(x)$

Domain is $(-\infty, 0) \cup (0, \infty)$, range is $\mathbb{R}$.

d) $h(x, y) = f(x) + g(y)$

Domain is $(-\infty, 0) \cup (0, \infty) \times \mathbb{R}$, range is $\mathbb{R}$.

e) $h(x, y) = (f(x), g(y))$

Domain is $\mathbb{R} \times (-\infty, 0) \cup (0, \infty)$, range is $[-1, \infty) \times (-\infty, 0) \cup (0, \infty)$.