Math 2013 - Introduction to Discrete Mathematics

Homework #5 - 2005.11.14

Due Date - 2005.11.28

Solutions

Consider the pairs of functions $f(n)$ and $g(n)$ given below, with $n \in \mathbb{N}$. Find values of $C > 0$ and $n_0 \in \mathbb{N}$ such that $f(n) = \mathcal{O}(g(n))$, i.e. find a $C > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$:

$$|f(n)| \leq C\,|g(n)|$$

1. Let $f(n) = 3n + 7$ and $g(n) = n$

   Let $n_0 = 1$. Notice that $f(1) = 10$, so if we choose $C = 10$, then we have the following:

   $$|3n + 7| = 3n + 7 \leq 10n, \quad \forall n \geq 1.$$

2. Let $f(n) = n^2 + 2n + 1$ and $g(n) = n^2$

   Notice that $n^2 + 2n + 1 = (n+1)^2$, so

   $$\left|n^2 + 2n + 1\right| = \left|(n+1)^2\right| = (n+1)^2, \quad \forall n \in \mathbb{N}$$

   Furthermore,

   $$(n+1)^2 \leq (n+n)^2 = 4n^2, \quad \forall n \in \mathbb{N}.$$

   Thus, choosing $n_0 = 1$ and $C = 4$ gives

   $$\left|n^2 + 2n + 1\right| \leq 4n^2, \quad \forall n \geq 1.$$

3. Let $f(n) = (n+1)^3$ and $g(n) = n^3$.
   This can be done similar to above and upon noting that $(n+1)^3 > 0, \quad \forall n \in \mathbb{N}$. So we have:

   $$(n+1)^3 \leq (n+n)^3 = 8n^3.$$

   So setting $C = 8$ and $n_0 = 1$, one has

   $$\left|(n+1)^3\right| \leq 8n^3, \quad \forall n \geq 1.$$

4. Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = n^2$.
   First, notice that $i > 0$ for $1 \leq i \leq n$. Thus

   $$\left|\sum_{i=1}^{n} i\right| = \sum_{i=1}^{n} i.$$

   Since $i \leq n$ for each term in the sum, we have the inequality

   $$\sum_{i=1}^{n} i \leq \sum_{i=1}^{n} n, \quad \forall n \in \mathbb{N}.$$

   However,

   $$\sum_{i=1}^{n} n = n^2.$$

Setting $n_0 = 1$ and $C = 1$, we have

$$\left| \sum_{i=1}^{n} i \right| \leq n^2, \quad \forall n \geq 1.$$