Math 2215 - Calculus 1

Homework #1 - 2005.08.24

Due Date - 2005.08.31

Solutions

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1. Show that $p(x) = a_0 + a_2x^2 + a_4x^4 + \ldots a_{2n}x^{2n}$ is even, where the a_j's are real.

   To show that $p(x)$ is even, we must show that $p(-x) = p(x)$. Setting $x \to -x$ in the above expression, one has:

$$p(-x) = a_0 + a_2(-x)^2 + a_4(-x)^4 + \ldots a_{2n}(-x)^{2n}.$$

   Notice that all the powers of $-x$ or of the form $2 \cdot k$, i.e. even.

$$(-x)^{2\cdot k} = \left((-x)^k\right)^2 = x^{2\cdot k}.$$

   Thus $p(-x) = p(x)$.

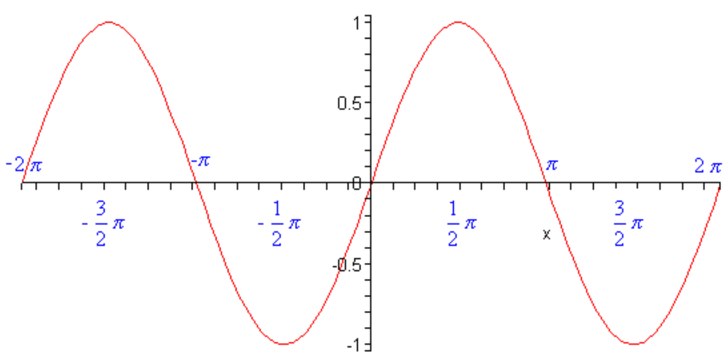
2. Given that $f(x) = x^2$ and $g(x) = \sqrt{x}$, compute the domains and ranges of the composition functions $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.

   First we will consider $f(g(x))$. Notice that g has domain $[0, \infty)$ and range $[0, \infty)$. Since f has domain $\mathbb{R}$, all elements a in the domain of g give values $g(a)$ in the domain of f. Thus the domain of $f \circ g$ is $[0, \infty)$. The range is also given by $[0, \infty)$.

   Next we will consider $g(f(x))$. The domain of f is once again $\mathbb{R}$, with range $[0, \infty)$, which is exactly the domain of g. So the domain of $g \circ f$ is $\mathbb{R}$ with the same range as that of g, which is $[0, \infty)$.

3. Compute the values of c such that $\sin(x + c)$ is an even function.

   Below is the graph of $\sin(x)$, which is an odd function.



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By shifting the graph to the right by $\frac{\pi}{(}$

Remember that the domain of $\ln(y)$ is $(0, \infty)$. Therefore, we require that $-x + 2 > 0$ which in turn implies that the domain of $L(x)$ must be $(-\infty, 2)$. The range is still $\mathbb{R}$.

5. Solve the expression $\ln(-y + 2) = x$ for y.

$$e^{\ln(-y+2)} = e^x$$

$$-y + 2 = e^x$$

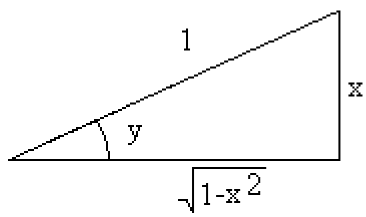
$$y = 2 - e^x$$

6. What is the relationship between your answer to problem 5 and problem 4?

The two functions referenced are inverses of each other on restricted domains.

7. Simplify the expression $\tan\left(\sin^{-1}(x)\right)$.

Setting $y = \sin^{-1}(x)$, we then have that $\sin(y) = x$. This gives us the following picture:



It is not easy to determine $\tan(y) = \frac{x}{\sqrt{1-x^2}}$.

8. Solve the equation $e^{2x-x^2} = \ln(4)$ for the variable x.

First we must take ln of both sides:

$$\ln\left(e^{2x-x^2}\right) = \ln\left(\ln(4)\right)$$

$$2x - x^2 = \ln\left(\ln(4)\right)$$

Putting everything on one side gives:

$$x^2 - 2x + \ln\left(\ln(4)\right) = 0$$

Notice now that this is now in the form $ax^2 + bx + c = 0$, with $a = 1$, $b = -2$ and $c = \ln\left(\ln(4)\right)$. This yields:

$$x = \frac{2 \pm \sqrt{4 - 4\ln\left(\ln(4)\right)}}{2}$$

which simplifies to:

$$x = 1 \pm \sqrt{1 - \ln\left(\ln(4)\right)}.$$

9. Consider the function $f(x) = x^4 + x^2 + 1$ with domain $[0, \infty)$ and range $[1, \infty)$. Find the inverse of $f(x)$ and state its domain and range. (*Hint: As an intermediary step, let* $z = x^2$.)

As the hint suggests, we will set $z = x^2$, which gives:

$$y = z^2 + z + 1$$

Solving for z requires that we move everything to one side:

$$z^2 + z + (1 - y) = 0.$$

This is of the form $az^2 + bz + c$, with $a = 1$, $b = 1$ and $c = 1 - y$. Solving for z gives:

$$z = -\frac{1}{2} \pm \frac{\sqrt{4y - 3}}{2}$$

Since $z = x^2$, we now have that

$$x = \pm\sqrt{-\frac{1}{2} \pm \frac{\sqrt{4y - 3}}{2}}.$$

Switching the variables x and y, we see that

$$f^{-1}(x) = \pm\sqrt{-\frac{1}{2} \pm \frac{\sqrt{4x - 3}}{2}}.$$

However, there is only one inverse, so me must determine which one is valid (of the 4 possible). We can immediately eliminate the possibility of the inverse having a negative sign in front of it, thus we now have:

$$f^{-1}(x) = \sqrt{-\frac{1}{2} \pm \frac{\sqrt{4x - 3}}{2}}.$$

Using the fact that $f^{-1}(1) = 0$, we see that

$$f^{-1}(x) = \sqrt{-\frac{1}{2} + \frac{\sqrt{4x - 3}}{2}}.$$

The domain of the inverse is the same as the range of the function, and the range of the inverse is the same as the domain of the function. The figure below shows the original function (blue), the line $y = x$ (black) and the inverse function (red).