Math 2215 - Calculus 1

Homework #2 - 2005.08.31

Due Date - 2005.09.09

Solutions

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1. Compute the following limit:

$$\lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4}$$

Notice that $x^3 - a^3 = (x-a)\left(x^2 + ax + a\right)$ and that $x^4 - a^4 = (x-a)(x+a)\left(x^2 + a^2\right)$. Therefore

$$\lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4} = \lim_{x \to a} \frac{(x-a)\left(x^2 + ax + a^2\right)}{(x-a)(x+a)(x^2 + a^2)},$$

and the $x - a$ terms cancel for x close, but not equal to a. So one can now evaluate:

$$\lim_{x \to a} \frac{(x-a)\left(x^2 + ax + a^2\right)}{(x-a)(x+a)(x^2 + a^2)} = \lim_{x \to a} \frac{\left(x^2 + ax + a^2\right)}{(x+a)(x^2 + a^2)} = \frac{3a}{4}.$$

2. Compute the following limit:

$$\lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

First get the numerator over a common denominator:

$$\lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h \to 0} \frac{\frac{2+h-2}{2(2+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{2(2+h)}}{h}.$$

Multiply top and bottom by $\frac{1}{h}$ to remove the h in the denominator. As a result, one can plug in $h = 0$ to get:

$$\lim_{h \to 0} \frac{1}{2(2+h)} = \frac{1}{4}.$$

3. Use the Squeeze Theorem to prove the following:

$$\lim_{x \to 0} \frac{x^2}{1 + (1 + x^4)^{\frac{5}{2}}} = 0$$

Hint: Notice that the function in question is always positive. What function can you bound it from above?

We have the following bounds:

$$0 \leq \frac{x^2}{1 + (1 + x^4)^{\frac{5}{2}}} \leq x^2$$

The result follows immediately upon use of the Squeeze Theorem.

4. Show that the function $p(x) = x\cos(\pi x) + x^2 - 1$ has at least one positive root and one negative root.

First notice that $p(0) = -1$. If we can find $a < 0$ and $b > 0$ such that $p(a) > 0$ and $p(b) > 0$, then the Intermediate Value Theorem can be applied. Notice that $p(2) = 5$ and $p(-1) = 1$. Thus a root must exist for some values c and

d where $-1 < c < 0$ and $0 < d < 2$.

5. What value should be assigned to a to make the function

$$g(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3, \end{cases}$$

continuous at $x = 3$?

First, we know that $\lim_{x \to 3^-} g(x) = 8$ and $\lim_{x \to 3^+} g(x) = 6a$. To make $g(x)$ continuous at $x = 3$, these two limits must be the same. Therefore, setting $6a = 8$ gives $a = \frac{4}{3}$.

6. Compute the following limit:

$$\lim_{x \to \infty} \frac{x\sqrt{4x^4 + 3x}}{12x^3 + 23x^2 - 16x + 4}$$

Here, letting x become large, the dominant term in the numerator is $x\sqrt{4x^4}$, and in the denominator, $12x^3$. Thus, as $x \to \infty$, we have that

$$\frac{x\sqrt{4x^4 + 3x}}{12x^3 + 23x^2 - 16x + 4} \sim \frac{x\sqrt{4x^4}}{12x^3} = \frac{1}{6}.$$

7. Let $f(x) = \frac{x-1}{x}$. Show that $f(x) \cdot f(1-x) = 1$.

First, we must compute $f(1-x)$:

$$f(1-x) = \frac{(1-x) - 1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1}$$

Next, we take

$$f(x) \cdot f(1-x) = \frac{x-1}{x} \cdot \frac{x}{x-1} = 1.$$